TMUA9. Model Answer + Analysis.

Cambridge Assessment Admissions Testing

TMUA/CTMUA **D513/01**

PAPER 1

November 2020 **75 minutes**

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

You **must** complete the answer sheet within the time limit.

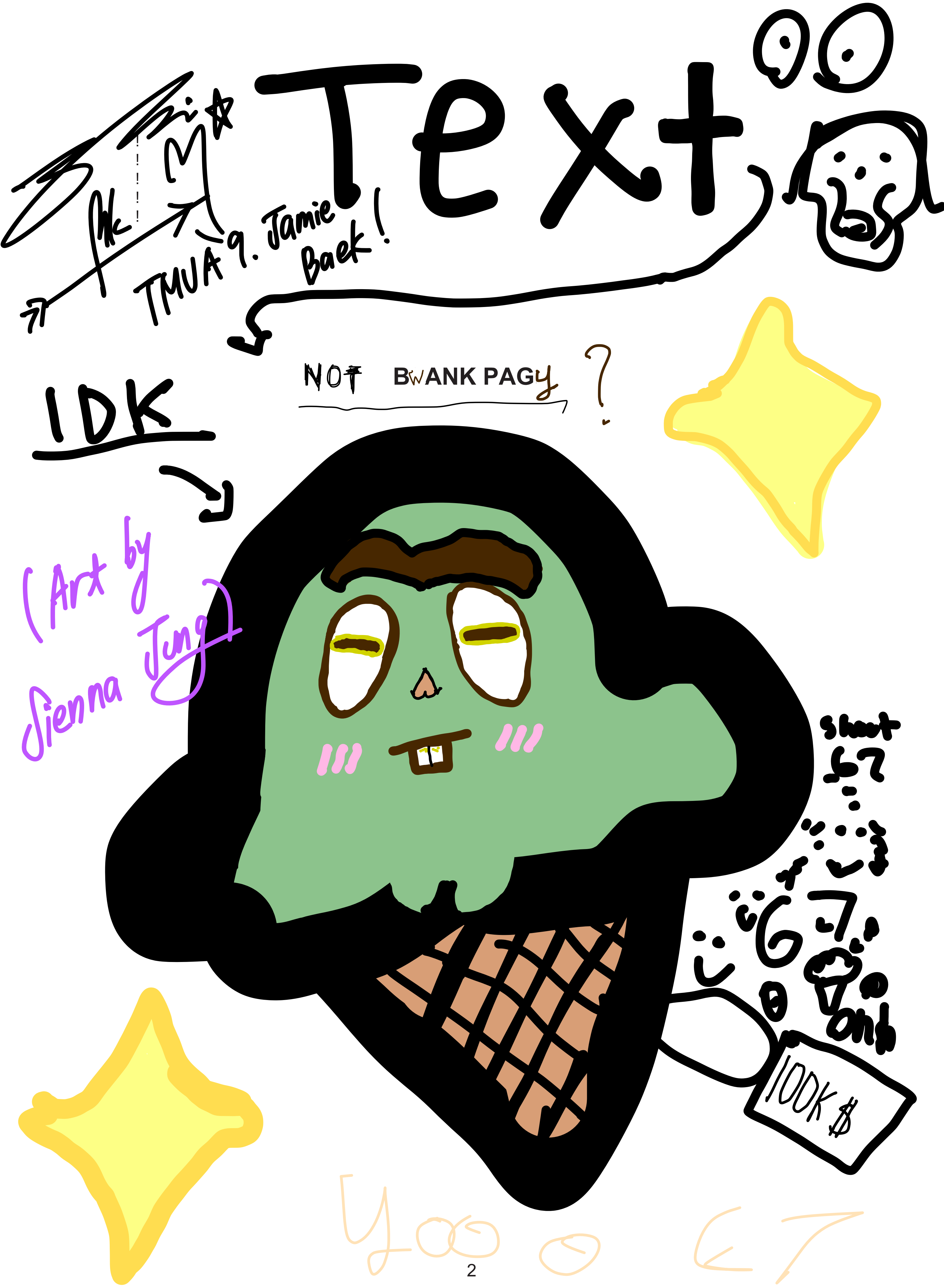
Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2


Text
TMUA 9. Jamie Baek!
IDK
NOT BWANK PAGY ?
(Art by Sienna Jung)
100K $

"Critical thought" = Don't use Quotient Rule.

Reaction Circuit = Simplify before Calculation

1 Which of the following is an expression for the first derivative with respect to x of

$$\frac{x^3 - 5x^2}{2x\sqrt{x}}$$

A $-\frac{\sqrt{x}}{2}$

B $\frac{\sqrt{x}}{4}$

C $\frac{3x - 5}{4\sqrt{x}}$

D $\frac{3\sqrt{x} - 5}{4\sqrt{x}}$

E $\frac{3\sqrt{x} - 10}{3\sqrt{x}}$

F $\frac{3x^2 - 10x}{3\sqrt{x}}$

$= (2x^{\frac{3}{2}})^{-1}(x^3 - 5x^2)$

$= \frac{1}{2}x^{-\frac{3}{2}}(x^3 - 5x^2)$

$f(x) = \frac{1}{2}(x^{\frac{3}{2}} - 5x^{\frac{1}{2}})$

$f'(x) = \frac{3}{4}x^{\frac{1}{2}} - \frac{5}{4}x^{-\frac{1}{2}}$

$= \frac{3x - 5}{4\sqrt{x}}$

Universal.

Simple Lesson). Seek Deletion/Optimisation before Automation (=Calculation).

See if we can perfect it / simplify methods before Calculation (= brute force algebra itself)

☆ Simple factor theorem:

A linear binomial $(x-c)$ is a factor of $f(x)$ if and only if $f(c)=0$.

2 $(2x+1)$ and $(x-2)$ are factors of $2x^3+px^2+q$

What is the value of $2p+q$?

A -10

B $-\frac{38}{5}$

C $-\frac{22}{3}$

D $\frac{22}{3}$

E $\frac{38}{5}$

F 10

$f(x)=2x^3+px^2+q$

$f(-\frac{1}{2})=f(2)=0.$

$f(-\frac{1}{2})=-\frac{1}{4}+\frac{1}{4}p+q=0.$

$-1+p+4q=0.$

$p+4q=1$ ... ①

$f(2)=16+4p+q=0.$

$4p+q=-16$ ... ②

① + ②: $5p+5q=-15$, $p+q=-3$.

$\therefore 3q=-2$, $q=-\frac{2}{3}.$

$3p=-19$, $q=\frac{-19}{3}.$

★ And ≡ $\cap$ (intersection).

★ or ≡ $\cup$ (Union).

3 Find the complete set of values of x for which

$$(x+4)(x+3)(1-x) > 0 \quad \text{and} \quad (x+2)(x-2) < 0$$

A $1 < x < 2$

B $-2 < x < 1$

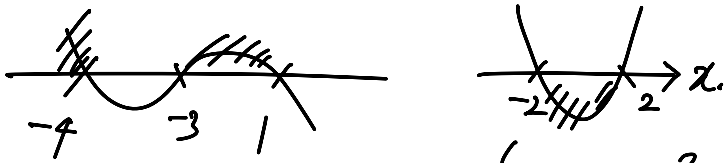
C $-2 < x < 2$

D $x < -2$ or $x > 1$

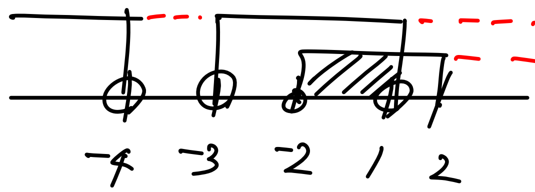
E $x < -4$ or $x > 2$

F $x < -4$ or $-3 < x < 1$

G $-4 < x < -2$ or $x > 1$



$\{x<-4 \text{ or } -3<x<1\} \cap \{-2<x<2.\}$.



$-2<x<+1$

★ Habituate sketching number line diagrams &

expressing inequalities.

✓ Tips) When doing so, the heights of all of the inequalities must be different.

→ If there are 3 different sets of Inequalities the heights must differ.

Sol x) Do not use the adamant $a_n = a+(n-1)d$, $b_n = br^{n-1}$ (Don't introduce both d/r in sequences if you can manage with it if not needed).

Arithmetic : $2a_n = a_{n-k} + a_{n+k}$

Geometric : $a_n^2 = a_{n-k} \cdot a_{n+k}$.

4 The 1st, 2nd and 3rd terms of a geometric progression are also the 1st, 4th and 6th terms, respectively, of an arithmetic progression.

The sum to infinity of the geometric progression is 12.

Find the 1st term of the geometric progression.

A 1

B 2

C 3

D 4

E 5

F 6

Sol 1) Use overall structure of geometric sequences.

$\underbrace{a \quad ar \quad ar^2}$.

$(ar-a) : (ar^2 - ar) = 3 : 2$.

$(\cancel{ar-a}) : r(\cancel{ar-a}) = 3 : 2$

$3r = 2$, $r = \frac{2}{3}$

$\frac{a}{1-r} = 3a = 12$

$a = 4$

Sol 2) Use overall structure of arithmetic sequences.

$a \quad a+3d \quad a+5d$

$a_n^2 = a_{n-1} a_{n+1}$ for geometric sequences.

$(a+3d)^2 = a(a+5d)$.

$\cancel{a^2} + \cancel{6ad} + 9d^2 = \cancel{a^2} + \cancel{5ad}$.

$ad + 9d^2 = 0$.

$a + 9d = 0$, $a = -9d$

$-9d \rightarrow -6d \rightarrow -4d$

$\left(\times \frac{2}{3}\right)$

$\frac{-9d}{1-\frac{2}{3}} = -27d = 12$.

$-9d = 4$

* Don't find d's actual value as d= ~. Just fulfill what is required.

5 The curve S has equation

$$y = px^2 + 6x - q$$

where p and q are constants.

S has a line of symmetry at $x = -\frac{1}{4}$ and touches the x-axis at exactly one point.

tangent to x-axis at one point.

What is the value of $p + 8q$?

A 6

B 18

C 21

D 25

E 38

$$y = p\left(x + \frac{1}{4}\right)^2 + 0$$

$$= p\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right)$$

$$= px^2 + \frac{1}{2}px + \frac{p}{16}$$

$$\frac{1}{2}p = 6, \quad p = 12$$

$$-q = \frac{p}{16} = \frac{12}{16}$$

$\times -8$ $\qquad$ $\times -8$

$$8q = \frac{12}{-2} = -6$$

$$p + 8q = 12 - 6 = 6.$$

* Substitution) the variable must be assigned a specific domain.

→ ACTION STEP) Even if it is $(-\infty, \infty)$, make a habit such that every single new variable introduced/assigned, write out domains next to it.

6 Find the maximum value of the function

$$f(x) = \frac{1}{5^{2x} - 4(5^x) + 7}$$

A $\frac{1}{7}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D 3

E 4

F 7

$$f(x) = \frac{1}{(5^x - 2)^2 + 3}$$

$(5^x > 0)$

$$\max(f(x)) = \min\{(5^x - 2)^2 + 3\}.$$

$$= 3 \text{ (at } 5^x = 2).$$

Check if domain includes the rest.

As $y=a^x$ is one-to-one function, thus if $f(a)\neq f(b)$ then $a\neq b$.

Command Term

Given bases are all perfect powers of 2.

Critical Thought

match out the bases all into 2.

7 Given that

$$2^{3x} = 8^{(y+3)}$$

and

$$4^{(x+1)} = \frac{16^{(y+1)}}{8^{(y+3)}}$$

what is the value of $x + y$?

A -23

B -22

C -15

D -14

E -11

F -10

$2^{3x} = 2^{3(y+3)}$

$\rightarrow x = y+3$

$2^{2(x+1)} = 2^{4(y+1)-3(y+3)}$

$\rightarrow 2(x+1) = 4y+4-3y-9$

$2x+2 = y-5$

$y = 2x+7$

$y = 2(y+3)+7$

$y = -13$

$x = y+3 = -10$

$x+y = -23$

* There was no need to compare what is bigger between -2 & p. → Only need to know the vertex's x values.

8 The function f is defined for all real x as

$$f(x) = (p - x)(x + 2)$$

Find the complete set of values of p for which the maximum value of $f(x)$ is less than 4.

$\max(f(x)) < 4$

A $-2 - 4\sqrt{2} < p < -2 + 4\sqrt{2}$

B $-2 - 2\sqrt{2} < p < -2 + 2\sqrt{2}$

C $-2\sqrt{5} < p < 2\sqrt{5}$

D $-6 < p < 2$

E $-4 < p < 0$

F $-2 < p < 2$

x-coordinate of vertex: $\frac{p-2}{2}$

$f\left(\frac{p-2}{2}\right) = \max(f(x))$ ($\because f(x)$ is a quadratic with negative leading coefficient)

$$f\left(\frac{p-2}{2}\right) = \left(p - \frac{p-2}{2}\right)\left(\frac{p-2}{2} + 2\right)$$

$$= \left(\frac{2p-p+2}{2}\right)\left(\frac{p-2+4}{2}\right)$$

$$= \left(\frac{p+2}{2}\right)^2$$

$$\left(\frac{p+2}{2}\right)^2 < 4 = 2^2$$

$$\left(\frac{p+2}{2}\right)^2 - 2^2 < 0$$

$$\left(\frac{p+2}{2} - 2\right)\left(\frac{p+2}{2} + 2\right) < 0$$

$$(p+2-4)(p+2+4) < 0$$

$$\underline{-6 < p < 2}.$$

☆ Vieta's formula is a must. (You must know at least for Quadratics & Cubics).

↳ Bro this is not that deep it's just such a simple simple concept that recurs, A LOT.

9 The quadratic expression $x^2 - 14x + 9$ factorises as $(x - \alpha)(x - \beta)$, where α and β are positive real numbers.

Which quadratic expression can be factorised as $(x - \sqrt{\alpha})(x - \sqrt{\beta})$?

A $x^2 - \sqrt{10}x + 3$

B $x^2 - \sqrt{14}x + 3$

C $x^2 - \sqrt{20}x + 3$

D $x^2 - 178x + 81$

E $x^2 - 176x + 81$

F $x^2 + 196x + 81$

$\left.\begin{array}{l} \alpha+\beta = 14 \\ \alpha\beta = 9 \end{array}\right\}.$

let this be $x^2 - mx + n$.

s.t $m = \sqrt{\alpha} + \sqrt{\beta} \longrightarrow \underbrace{\alpha+\beta}_{14} + 2\underbrace{\sqrt{\alpha\beta}}_{} = m^2$

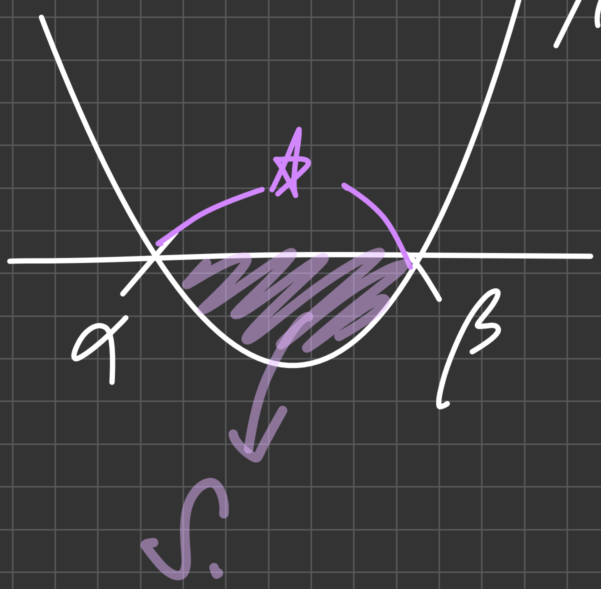
$n = \sqrt{\alpha\beta} = \sqrt{9} = 3$

$14 + 2\times 3 = 20 = m^2$.

$m = 2\sqrt{5}$

9. Extension to Vieta's Formulae.

$f(x) = a(x-\alpha)(x-\beta)$.



* Quick Recap: $S = \frac{1}{6}|a| \cdot ☆^3$ s.t. $☆ = |\beta - \alpha|$.

But In Vieta's formula, is there a more generalised Algebraic Approach for expressing α, β in terms of coefficients?

↓

let $ax^2+bx+c = a(x-\alpha)(x-\beta)$

From Vieta's Formula (Again, as always, it's not that deep... just expand & compare coefficients)

↓

$$\begin{cases} \alpha+\beta = -\frac{b}{a} \\ \alpha\beta = \frac{c}{a} \end{cases}$$

⇒ But... How about $\alpha-\beta$? Is there a general form for $\alpha-\beta$? (We aren't taught this in A-levels.)

In a nutshell, $|\beta-\alpha| = \frac{\sqrt{D}}{|a|}$

Let's go back to the core.

$ax^2+bx+c=0$'s two roots are α and β.
$(a \neq 0,\ a, b, c \in \mathbb{R})$ & $\beta > \alpha$

Lets assume $D \geq 0$ thus $\alpha, \beta \in \mathbb{R}$ (May be distinct/repeated).

$$a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c$$

I'm Just proving Quadratic Formula here. (Complete squares).

$$= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c=0$$

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

(Discriminant).

$$x+\frac{b}{2a}=\frac{\pm\sqrt{D}}{2a} \quad \text{s.t } D=b^2-4ac$$

$$x=\frac{-b+\sqrt{D}}{2a},\ \frac{-b-\sqrt{D}}{2a}$$

As $\beta > \alpha$, $\beta=\frac{-b+\sqrt{D}}{2a}$, $\alpha=\frac{-b-\sqrt{D}}{2a}$

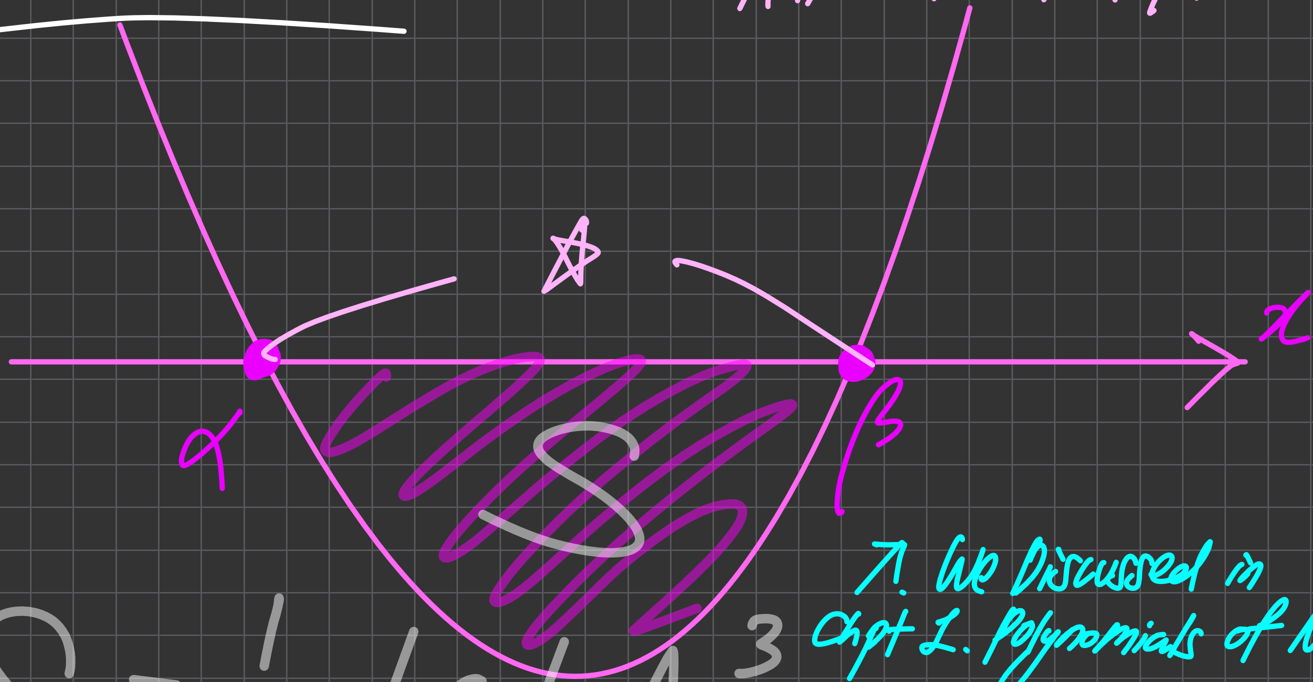
(I Just chose to say so)

$$\alpha+\beta=\frac{-b+\sqrt{D}}{2a}+\frac{-b-\sqrt{D}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

$$|\beta-\alpha|=\left(\frac{-b+\sqrt{D}}{2a}\right)-\left(\frac{-b-\sqrt{D}}{2a}\right)=\frac{2\sqrt{D}}{2a}=\frac{\sqrt{D}}{a}$$

9. Coalescence.

$f(x) = a(x-\alpha)(x-\beta)$



$$S = \frac{1}{6}|a|\star^3$$

↗. We Discussed in Chpt 1. Polynomials of NOZR why this frame is key.

*** Previously, we proved that $\beta - \alpha = \dfrac{\sqrt{D}}{|a|}$

As $\star = |\beta - \alpha|$ = Distance between THE X-COORDINATES

↓

Substitute $\star = \dfrac{\sqrt{D}}{|a|}$;

$$S = \frac{1}{6}|a| \cdot \left\{\frac{\sqrt{D}}{|a|}\right\}^3 = \frac{1}{6}|a| \cdot \frac{D\sqrt{D}}{|a|^3} = \frac{D^{\frac{3}{2}}}{6a^2}$$

Not Compulsory to memorise, Just to show the beauty of connecting S(Area), a(leading coefficient), and D (Discriminant.)

① Do all the translations/transformations sequentially & seperately.

② THEN. Only then after finishing several translations can you substitute back into the function.

9. Translations.

10 The following sequence of transformations is applied to the curve $y = 4x^2$

1. Translation by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$
2. Reflection in the x-axis
3. Stretch parallel to the x-axis with scale factor 2

What is the equation of the resulting curve?

A $y = -x^2 + 12x - 31$

B $y = -x^2 + 12x - 41$

C $y = \quad x^2 + 12x + 31$

D $y = \quad x^2 + 12x + 41$

E $y = -16x^2 + 48x - 31$

F $y = -16x^2 + 48x - 41$

G $y = \quad 16x^2 - 48x + 31$

H $y = \quad 16x^2 - 48x + 41$

$f(x, y) = 0$

$f(x-3, y+5) = 0.$

* notice how I did not $-(y-5)$, so I did not multiply (-1) to the entire bit, only the y itself.

$f(x-3, -y+5) = 0.$

$f(\underbrace{\tfrac{1}{2}x-3}_{X}, \underbrace{-y+5}_{Y}) = 0.$

$(-y+5) = 4\left(\tfrac{1}{2}x-3\right)^2$

$y-5 = -(x-6)^2.$

$y = -(x-6)^2+5$

$= -x^2+12x-36+5$

$= -x^2+12x-31$

9. Translation.

$T_1, T_2 \cdots T_n$ }.

General principle) For transforming a graph, express it sequentially, compute the transformations sequentially. THEN, ONLY THEN do you finally substitute all of it.

(Know it's coming from graphical transformation, but by all means, when actually solving them the step of visualisation is not needed.)

e.g) $\underset{X}{\underbrace{f(x}}, \underset{Y}{\underbrace{y)}} = 0 \xrightarrow{T_n} f(\underset{X}{\underbrace{17-2y}}, \underset{Y}{\underbrace{(x+42)}}) = 0 \xrightarrow{T_{n+1}} g(x)$.

let T_2 (2nd transformation be: (Stretch. parallel to x-axis by scale factor 3. translation in $+y$-axis direction by 2

find $g(x)$.

$f(\underset{X}{\underbrace{17-2(y-2)}}, \underset{Y}{\underbrace{\tfrac{1}{3}x+42}}) = g(x)$.

So only substitute the translation into the variable itself, not the expression around it. }.

(examples)

$$f_0(\overset{X}{x},\overset{Y}{y})=0$$

T_1

$$f_1(x,y)=0 \rightarrow f_0(x-2,y-4)=0.$$

$$f_n\left(e^{\pi^{e^{\pi}}}-4\overset{y-2}{y},\ 24\overset{\frac{1}{3}x}{x}-72\right).$$

stretch parallel to x-axis by 3.
translate parallel to y-axis by $\bar{2}$.

T_2

T_3

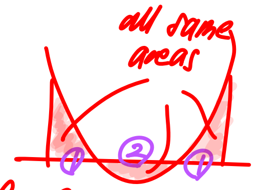
$\vdots$

T_n

$$f_n(x,y)=0.$$

THEN! We Substitute.

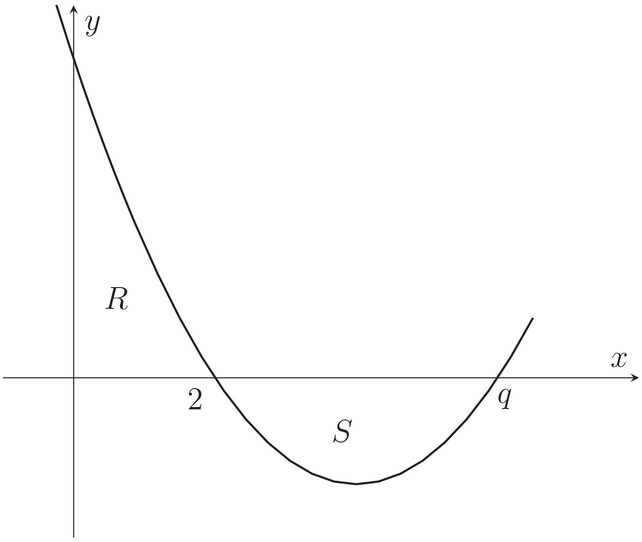
Repeated in NOIR.



for Quadratics.

If you knew this, this question takes 1 second to solve.

11 The quadratic function shown passes through $(2, 0)$ and $(q, 0)$, where $q > 2$.



What is the value of q such that the area of region R equals the area of region S?

A $\sqrt{6}$

B 3

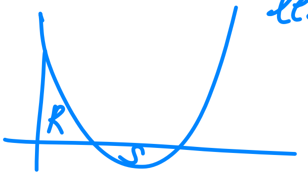
C $\frac{18}{5}$

D 4

E 6

F $\frac{33}{5}$

(If you did not know Ratios). Sol 1)



let $f(x) = a(x-2)(x-p)$

$a \neq 0,\ a \in \mathbb{R}^+$

$R = \int_0^2 f(x)\,dx$, $S = -\int_2^p f(x)\,dx$ (under x-axis).

$R - S = \int_0^2 f(x)\,dx + \int_2^p f(x)\,dx = 0$. $(\because R = S)$.

$RHS = \int_0^p f(x)\,dx = \int_0^p a(x-2)(x-p)\,dx = 0$

$a \neq 0$ so $\int_0^p (x-2)(x-p)\,dx = 0$.

Expand & integrate & algebra.

☆ When solving equations & you decide to solve it using graphs ORDER matters.

☆ focus on points on the integer grid.

12 How many real solutions are there to the equation

$$3\cos x = \sqrt{x}$$

$\sqrt{x} \geq 0$

$x \geq 0$.

where x is in radians?

A 0

B 1

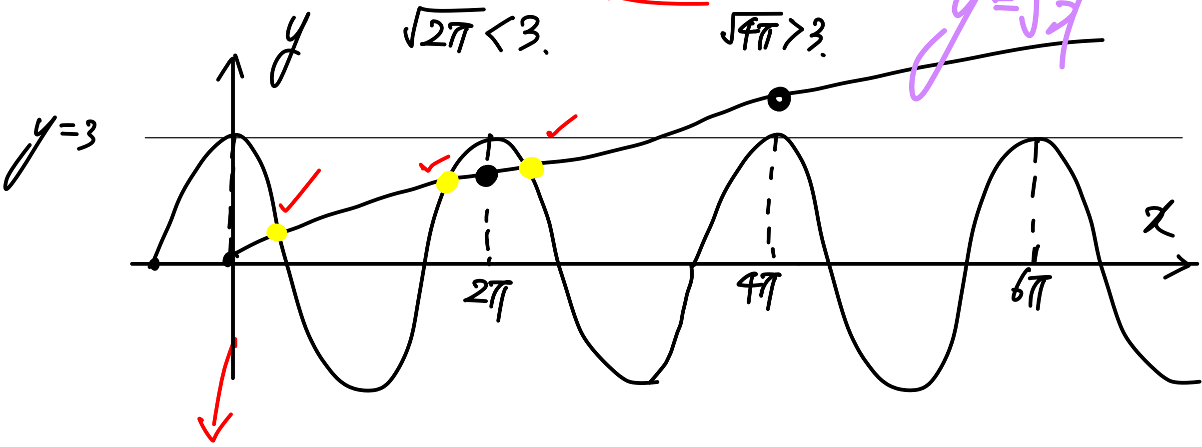
C 2

D 3

E 4

F 5

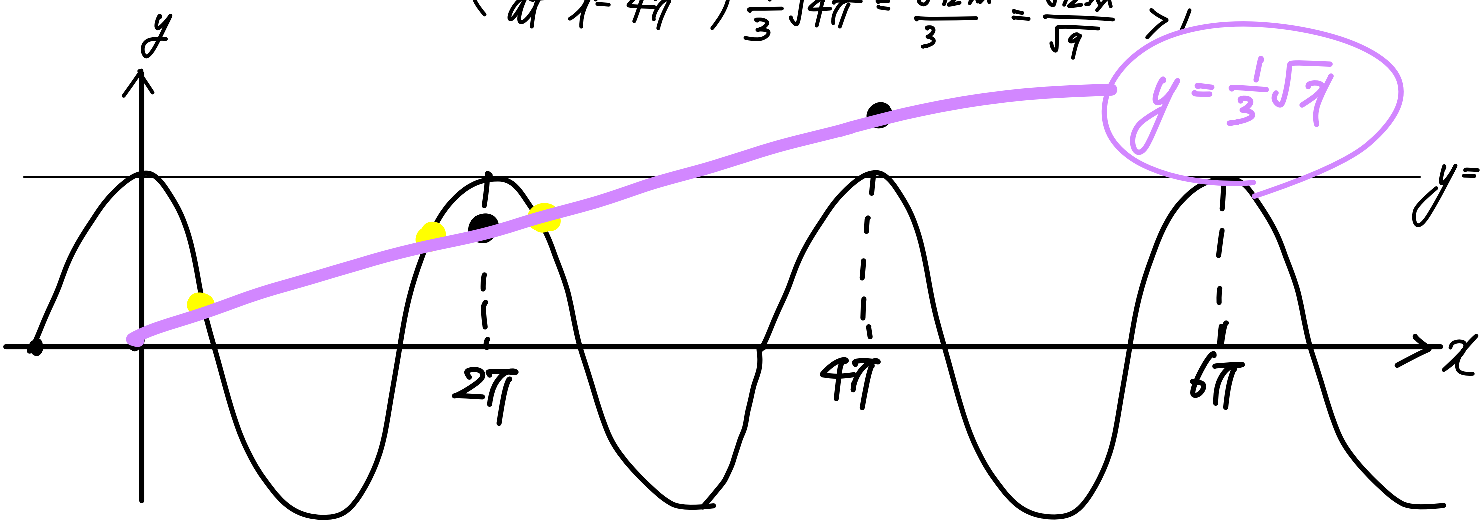
G infinitely many



my key points of analysis were points s.t. y values were integers specifically local maximum.

Sol 2) $\cos x = \frac{1}{3}\sqrt{x}$

$$\begin{cases} \text{at } x = 2\pi \,) & \frac{1}{3}\sqrt{2\pi} = \frac{\sqrt{6.xx}}{3} = \frac{\sqrt{6.xx}}{\sqrt{9}} < 1 \\ \text{at } x = 4\pi \,) & \frac{1}{3}\sqrt{4\pi} = \frac{\sqrt{12.xx}}{3} = \frac{\sqrt{12.xx}}{\sqrt{9}} > 1 \end{cases}$$



9. Recent TMUA trends.

9. The Complete Algorithm for … Sketching composite functions!

↑ Lectures on Composite Functions.

① $f(x)$

Objective is to find key points you want to observe the graph's estimate around & Do so.

(1) Domain & Range. (The most fundamental thing about functions)
X Y

(2) X-intercept, Y-intercept, & points excluded from the domain.

(3) Asymptotes. (Asymptotes are usually formed around x-coordinates of values excluded from Domain.

(4) End behaviour: ($x \to \pm\infty$, $x \to a$ s.t. a are the x values of vertical asymptotes)

Practice this as a tickbox you keep moving forward.

② $f'(x)$

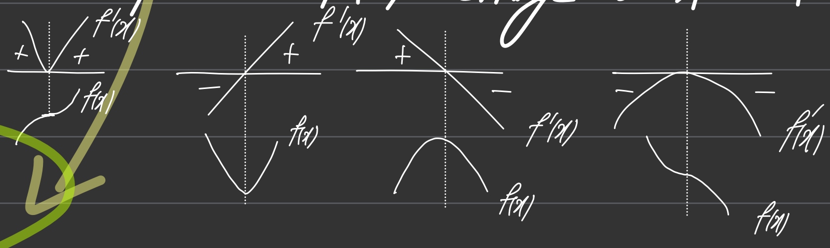
Objective is to observe how $f(x)$'s graph changes, based on key points (stationary points s.t. $f'(x)=0$).

(1) $f'(x)=0 \longrightarrow$ Stationary points.

(2) let $\alpha_1, \alpha_2, \cdots \alpha_n$ be stationary points : $x \to \alpha+$, $x \to \alpha-$

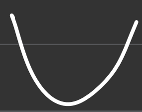
HOW DOES THE SIGN OF $f'(x)$ change as $x \to \alpha$?

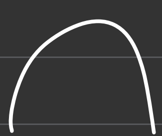
(Either 1 of 4)



③ $f''(x)$.

(1) Concavity.

$f''(x) \geq 0 \rightleftharpoons$  shape (Concave Up) ($\equiv$ Convex)

$f''(x) < 0 \rightleftharpoons$  shape (Concave Down) ($\equiv$ Concave)

In common sense, Just curvature!

7. TMUA 2020 Paper 2 Model Answer + Analysis.

* for this question, it's not necessary AT ALL, but as recent trends necessitate the ability to sketch composite functions... let's try so!
(Predicated on the previous algorithm)

$3\cos x = \sqrt{x}$

$\boxed{\text{(Sol 1)}\quad 3 = \dfrac{\sqrt{x}}{\cos x}}$

let $f_1(x) = \dfrac{\sqrt{x}}{\cos x}$, so we'll compare two graphs $\begin{cases} y = f_1(x) \\ y = 3 \end{cases}$

[f(x)]

Heuristics for X & Y selection is excluding which inequality the x (input) and y (output) can't have.

① X & Y : X: $x \geq 0$. (x can't be negative thus non-negative)
Y: $(-\infty, \infty)$

** But! $\cos x \neq 0$ ($\because$ Denominator $\neq 0$, so $\cos x \neq 0$ thus $x \neq \frac{2n+1}{2}\pi$ ($n \in \mathbb{Z}$)

② x & y intercept at $x = 0$

③ Asymptotes
$x \to \frac{1}{2}\pi + : \dfrac{\oplus}{0-} = -\infty$
$x \to \frac{1}{2}\pi - : \dfrac{\oplus}{0+} = +\infty$

Generalised into...
$x \to \frac{1}{2}(2n+1)\pi + \;:\; -\infty$
$x \to \frac{1}{2}(2n+1)\pi - \;:\; +\infty$

④ End Behaviour:
$\lim_{x \to \infty} \dfrac{\sqrt{x}}{\cos x} = \dfrac{+\infty}{\text{Altering signs}}$: Does not converge = Diverge.

[f'(x)].

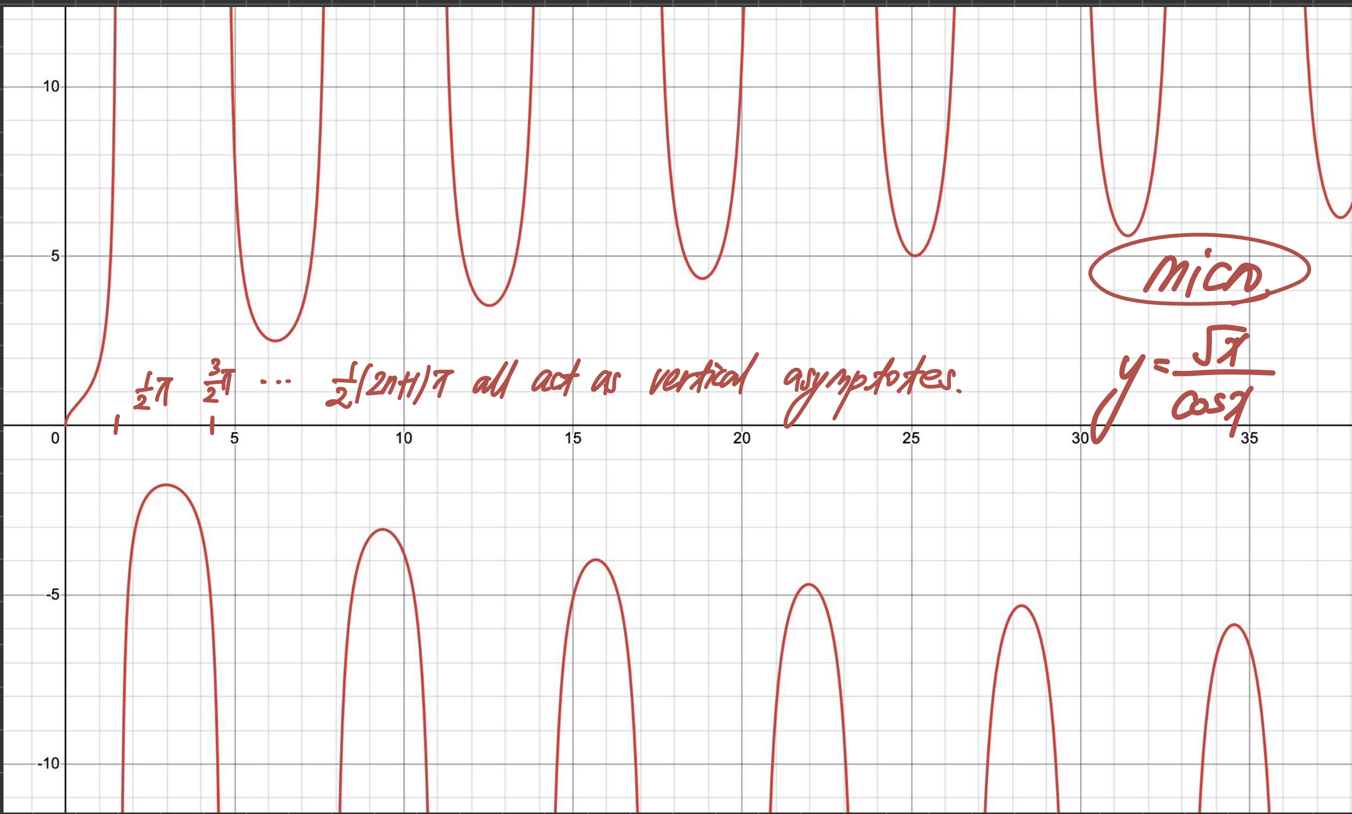
Simple Product Rule.

$\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{\cos x}\right) = \left(\sqrt{x} \cdot \dfrac{1}{\cos x}\right)' = \dfrac{1}{2\sqrt{x}} \cdot \cos x + \sqrt{x} \times \dfrac{-\sin x}{\cos^2 x}$

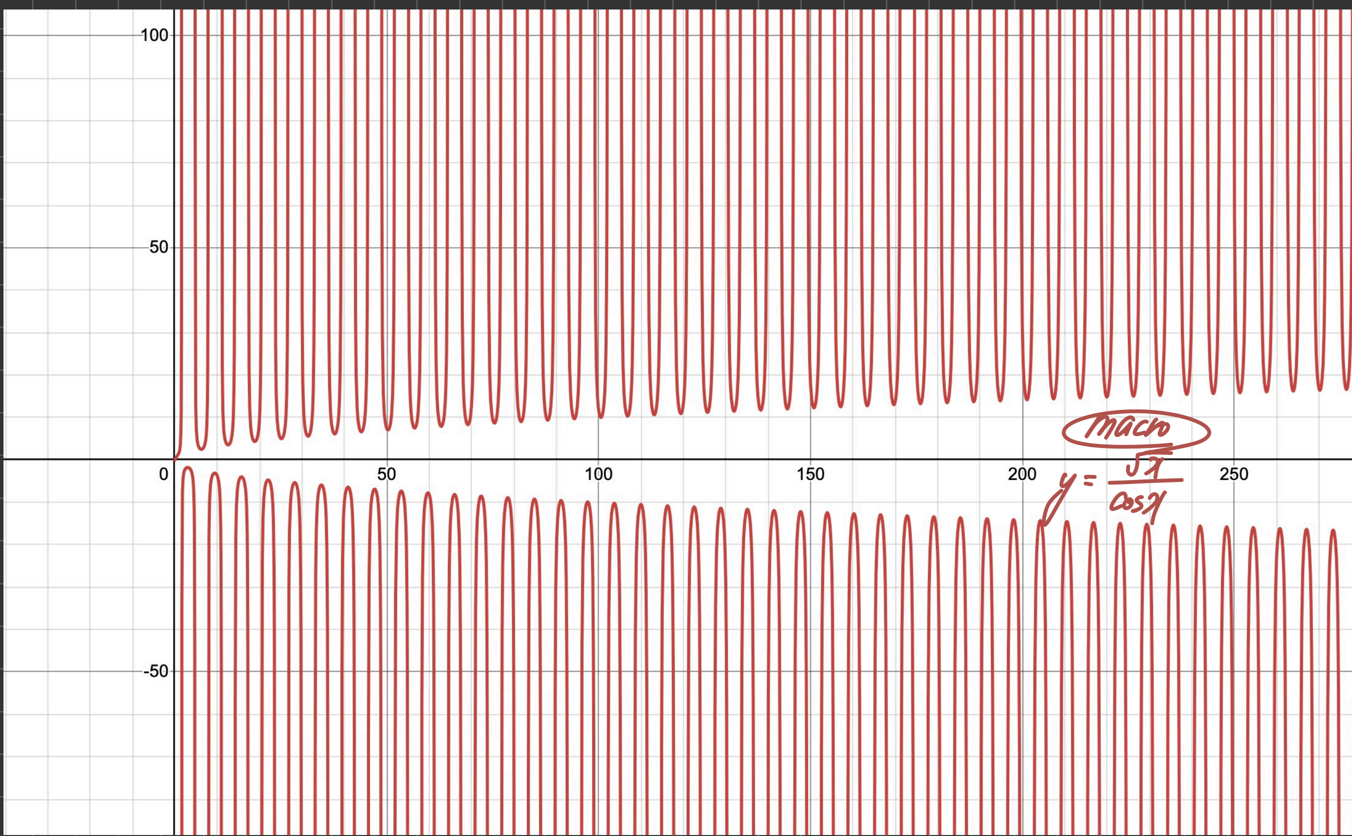
$= \dfrac{\cos^3 x - \sqrt{x}\sin x}{2\sqrt{x} \times \cos^2 x}$

If you were to pinpoint it...

Numerator $= 0$.



micro
$y = \frac{\sqrt{x}}{\cos x}$
$\frac{1}{2}\pi$ $\frac{3}{2}\pi$ ⋯ $\frac{1}{2}(2n+1)\pi$ all act as vertical asymptotes.
10
5
-5
-10
0
5
10
15
20
25
30
35



macro
$y = \frac{\sqrt{x}}{\cos x}$
100
50
-50
0
50
100
150
200
250

$$\boxed{\text{sol 2)} \quad 3\cos x = \sqrt{x}}$$

sol 2) $\dfrac{\cos x}{\sqrt{x}} = \dfrac{1}{3}$

let $f_2(x) = \dfrac{\cos x}{\sqrt{x}}$, we'll compare $\begin{cases} y = f_2(x) \\ y = \frac{1}{3} \end{cases}$

[$f(x)$]

① X & Y (As always domain & range).

$X: x \mid x \in (-\infty, 0), (0, \infty)$.

$Y: y \in \mathbb{R}$.

② Intersections with x-axis at $\cos x = 0$ ($\because \sqrt{x} \neq 0$)

As $\sqrt{x} \neq 0$, $\dfrac{\cos x}{\sqrt{x}} = 0$; $\cos x = 0$ at $x = \frac{1}{2}(2n+1)\pi$ s.t $n \in \mathbb{Z}$.

③ Asymptotes. ($x \neq 0$.) $\rightarrow \begin{cases} x \to 0+ & \frac{1}{0+} = +\infty. \text{ (Vertical Asymptote).} \\ x \to 0- & \text{Doesn't exist as not included in Domain.} \end{cases}$

↳ Did not need to be analysed.

[$f'(x)$]

$f(x) = \dfrac{\cos x}{\sqrt{x}} = x^{-\frac{1}{2}} \cdot \cos x$.

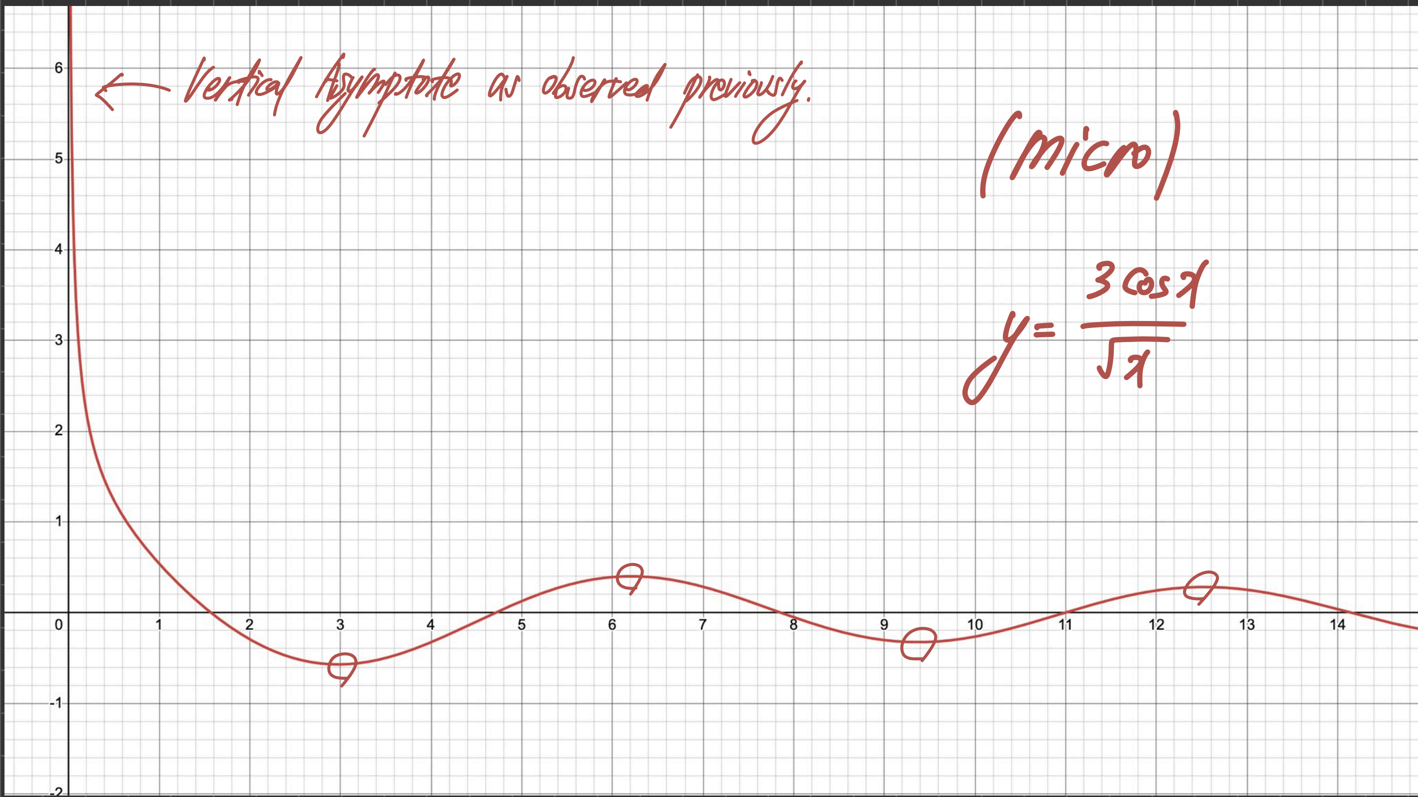
$f'(x) = -\frac{1}{2} x^{-\frac{3}{2}} \cdot \cos x + x^{-\frac{1}{2}} \cdot -\sin x$

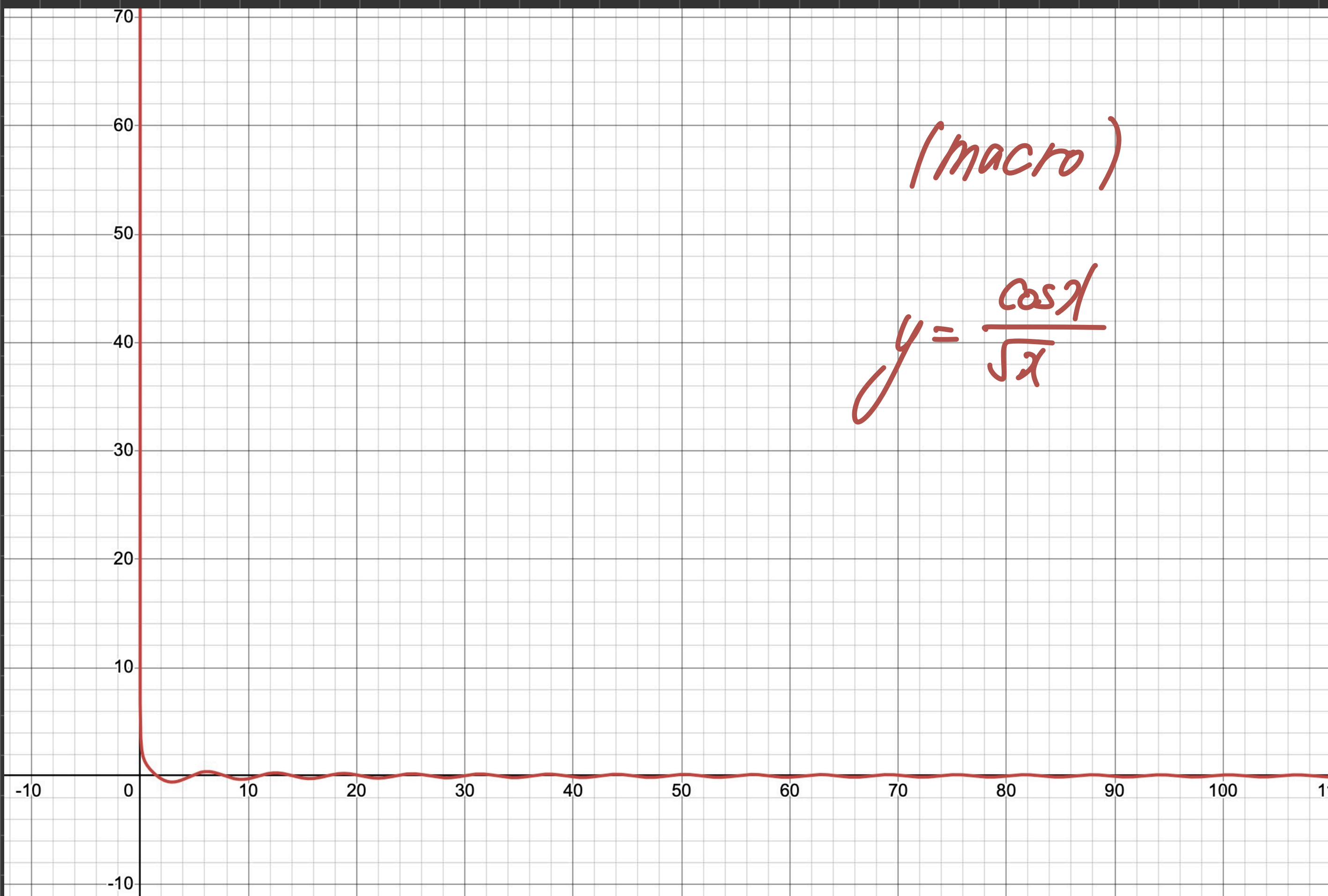
$= -x^{-\frac{1}{2}}\left(\frac{1}{2x}\cos x + \sin x\right) = 0$ at $x = \infty$ & $\frac{1}{2x}\cos x + \sin x = 0$

↓ $\frac{1}{2x} = -\tan x$.

↓ Can't solve this algebraically...

It's enough to know that The roots here are the stationary points.


Vertical Asymptote as observed previously.
(micro)
$y = \frac{3\cos x}{\sqrt{x}}$


(macro)
$y = \frac{\cos x}{\sqrt{x}}$

9. sketching reciprocal functions.

I mention this because Sol(1) sketched $\frac{\sqrt{x}}{\cos x}$ and Sol(2) sketched $\frac{\cos x}{\sqrt{x}}$,,, so this blackboard presents a general algorithm to sketch $\frac{1}{f(x)}$ from $f(x)$, and vice versa.

↗ TMUA9 sketching Lectures.

< Sketch $\{f(x)\}^{-1}$ using $f(x)$'s graph >

not inverse, Just $\frac{1}{f(x)}$ (= Reciprocal function)

① find values such that $|f(x)|=1$ $(f(x)=\pm 1)$

(you may ask... why? It's because solving $f(x)=\frac{1}{f(x)}$ $(f(x)\neq 0)$ implies
$\{f(x)\}^2-1=0$
$(f(x)-1)(f(x)+1)=0$ So
$f(x)=\pm 1$.

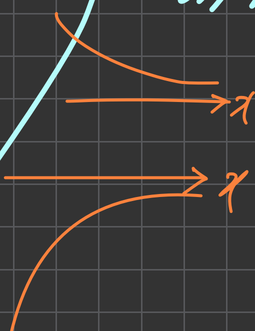
and set those as fixed points.

② find x intercepts of $y=f(x)$ (= points s.t. $f(x)=0$) & Determine left/right sided sign of asymptotes.

Change $\begin{pmatrix} \text{increasing} \Rightarrow \text{decreasing} \\ \text{decreasing} \Rightarrow \text{increasing} \end{pmatrix}$ for certain intervals.

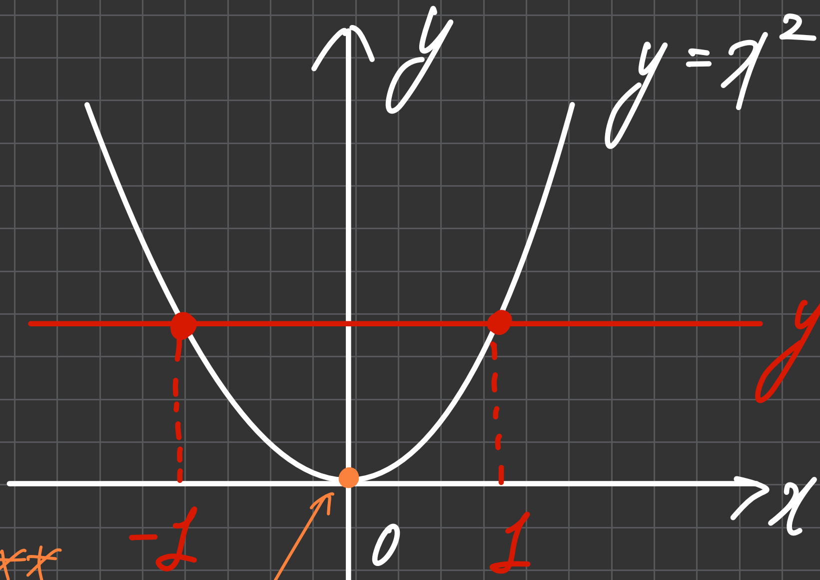
(Algebraic Proof on next page.

* If $f(x) \to +\infty$, $\frac{1}{f(x)} \to 0$ from up
  If $f(x) \to -\infty$, $\frac{1}{f(x)} \to 0$ from down



* If $f(x) \to 0+$, $\frac{1}{f(x)} \to +\infty$
  If $f(x) \to 0-$, $\frac{1}{f(x)} \to -\infty$
  If $|f(x)|=1$, $\frac{1}{|f(x)|}=1$

e.g) find $y=\frac{1}{x^2}$ using $y=x^2$.



$y=1$ (fix points first)

*** find x-intercepts.

0+ is an informal way of saying approaching 0 from the right, thus a positive value.

↕

0− is a negative value.

As $x \to 0^+$, $x^2 \to 0^+$

$x \to 0^-$, $x^2 \to 0^+$

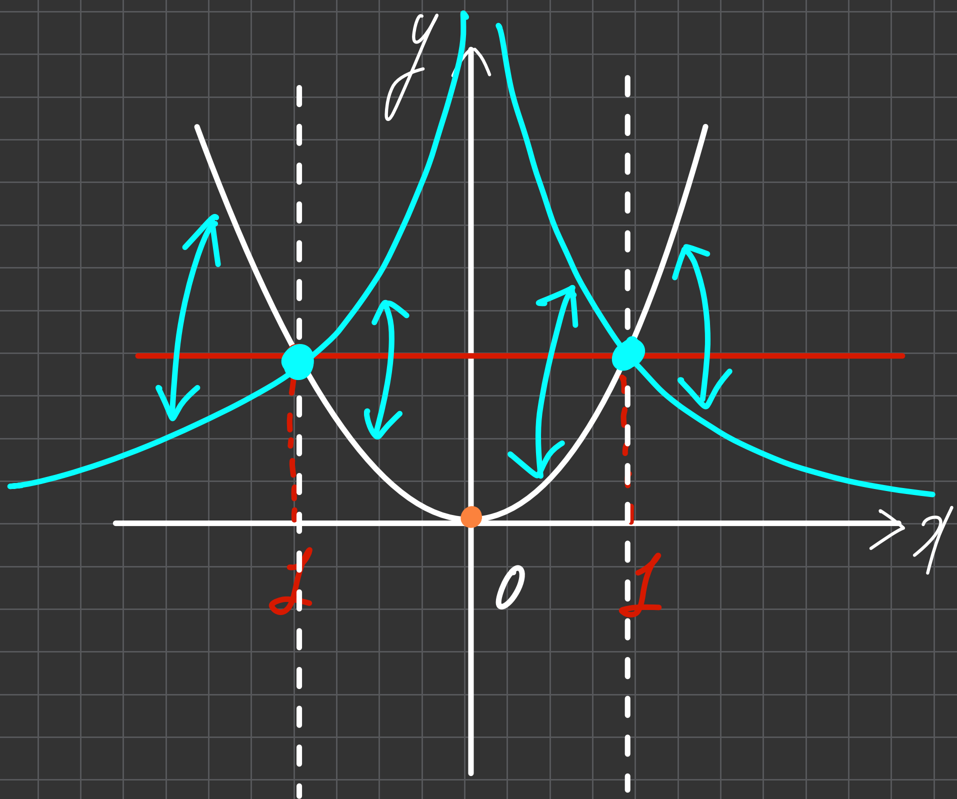
thus $\lim_{x\to 0^+} \frac{1}{x^2} = \frac{1}{0^+} = +\infty$

Hence $\lim_{x\to 0^-} \frac{1}{x^2} = \frac{1}{0^+} = +\infty$

Both 0+, 0− are not actually 0, so they can go into denominators.

$x<0 \to y=x^2$ Decreases. $\to y=\frac{1}{x^2}$ Increases.

$0<x \to y=x^2$ Increases. $\to y=\frac{1}{x^2}$ Decreases.



9. Why $f(x)$ and $\frac{1}{f(x)}$ has opposite signs for their first Derivatives,

$\downarrow \frac{d}{dx}$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \underbrace{\frac{-f'(x)}{\{f(x)\}^2}}$$

$\left(\frac{1}{\{f(x)\}^2} \geq 0\right.$, so does not impact overall sign of $f'(x)$ at all).

$\downarrow$

Hence, $f'(x)$ has opposite signs with $\frac{d}{dx}\left(\frac{1}{f(x)}\right)$ $\left(\equiv \frac{-f'(x)}{\{f(x)\}^2}\right)$

$\downarrow$ Graphically

$\boxed{1}$ $y = f(x)$ increases at $[a,b]$ $\rightleftharpoons$ $y = \frac{1}{f(x)}$ decreases at $[a,b]$ } Graphically

$\left(\equiv \frac{d}{dx}(f(x)) \geq 0\right)$ $\left(\equiv \frac{d}{dx}\left(\frac{1}{f(x)}\right) \leq 0\right)$ Algebraically.

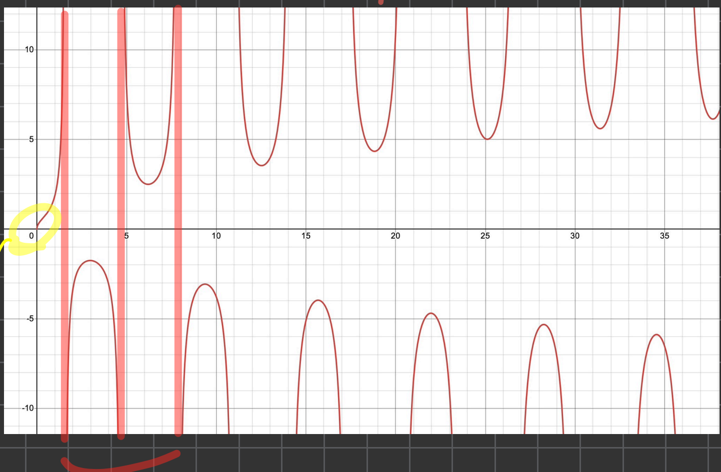
$\boxed{2}$ $y = f(x)$ decreases at $[a,b]$ $\rightleftharpoons$ $y = \frac{1}{f(x)}$ increases at $[a,b]$ } Likewise graphically.

$\left(\equiv \frac{d}{dx}(f(x)) \leq 0\right)$ $\left(\equiv \frac{d}{dx}\left(\frac{1}{f(x)}\right) \geq 0\right)$ Algebraically.

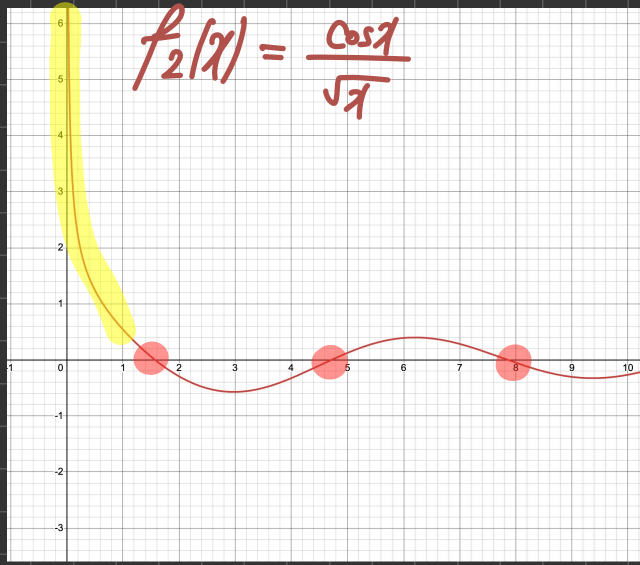
Applications.

Can we sketch $f_2(x)$ if we knew $f_1(x)$? AND Vice Versa; Can we sketch $f_1(x)$ if we knew $f_2(x)$'s graph?

$$f_1(x) = \frac{\sqrt{x}}{\cos x}$$



$$f_2(x) = \frac{\cos x}{\sqrt{x}}$$

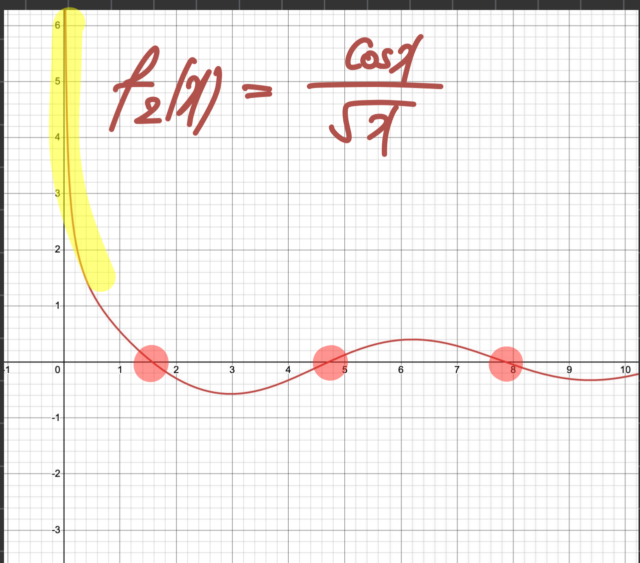


identical x-coordinates s.t. $f_2(x)=0$.

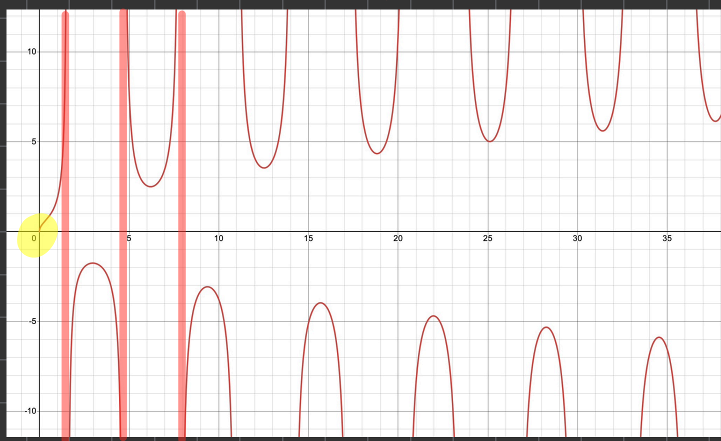
x Coordinates s.t $f_1(x)$ has vertical asymptotes $\rightleftarrows$ x Coordinates s.t $f_2(x)$ has x-intercept.

x Coordinate s.t $f_2(x)$ has vertical asymptotes $\rightleftarrows$ x Coordinates s.t $f_1(x)$ has x-intercepts. (in this case simply origin).

$$f_2(x) = \frac{\cos x}{\sqrt{x}}$$



$$f_1(x) = \frac{\sqrt{x}}{\cos x}$$



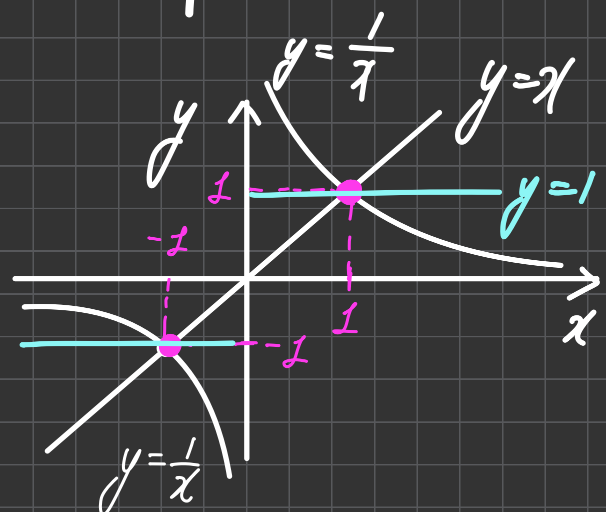
Examples (further)

① $y = \sin x \longrightarrow y = \csc x$

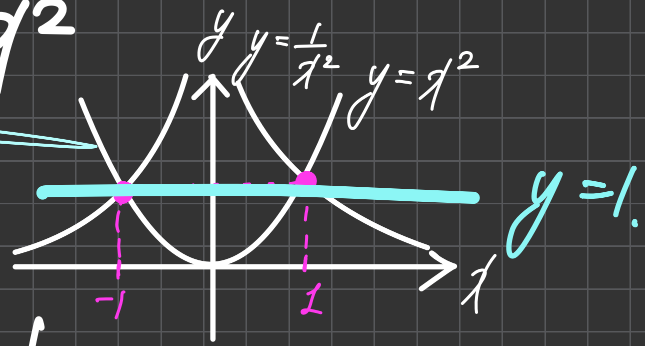
② $y = \cos x \longrightarrow y = \sec x$

③ $y = \tan x \longrightarrow y = \cot x$

$* \; y = x \longrightarrow y = \frac{1}{x}$

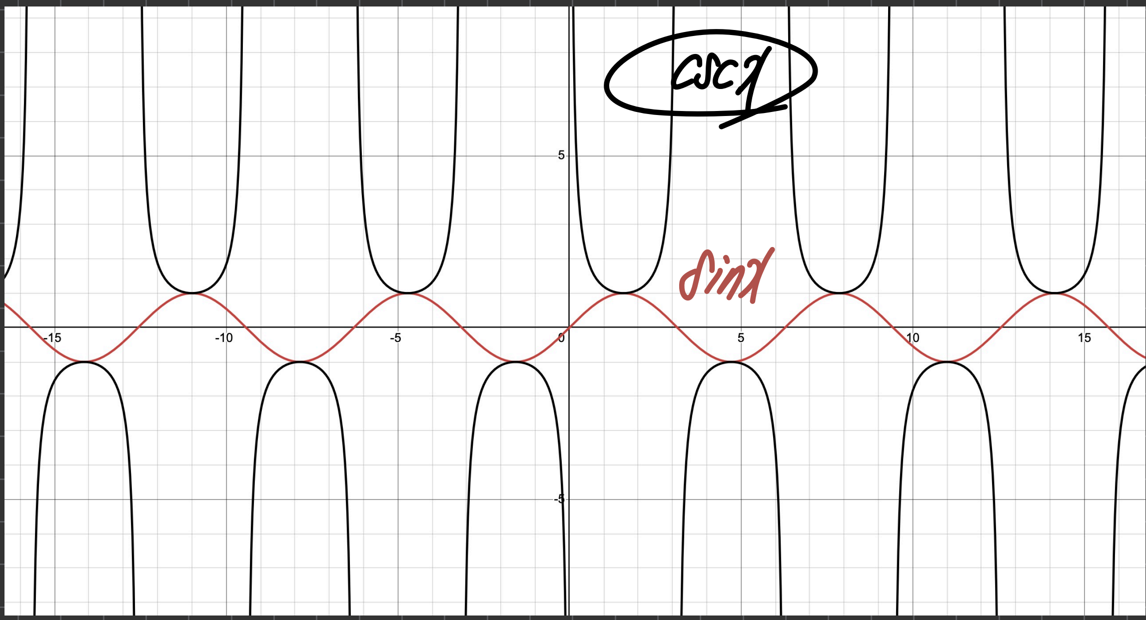


$* \; y = x^2 \rightarrow y = \frac{1}{x^2}$

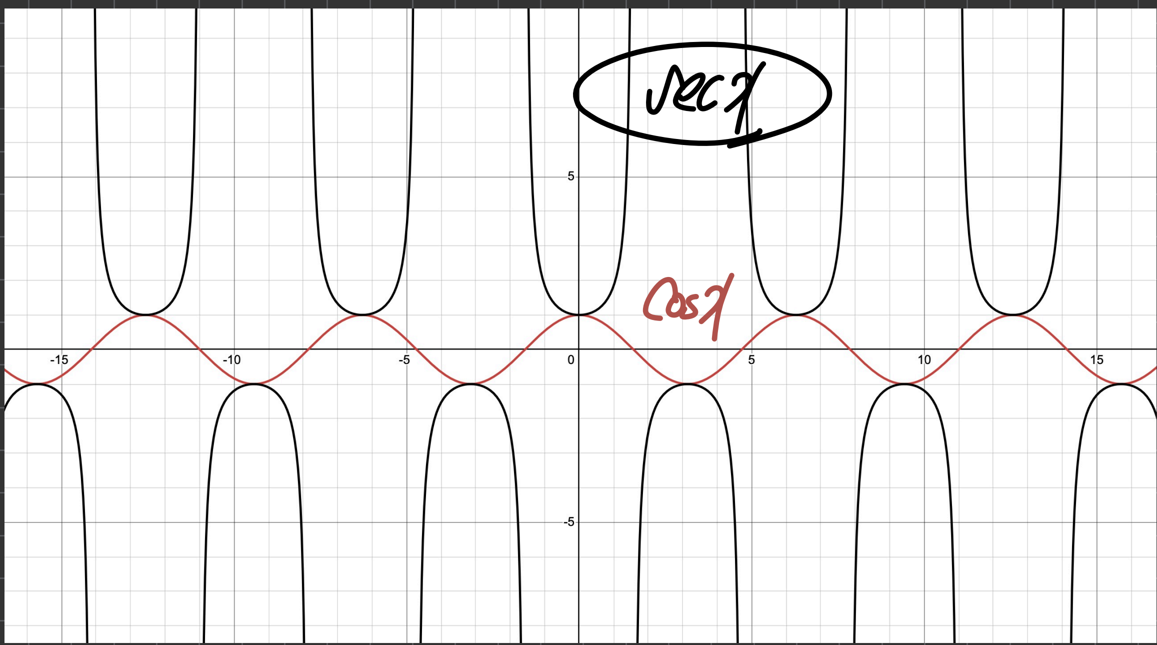


$y = x^{2k} \rightarrow y = \frac{1}{x^{2k}}$ (s.t $k \in \mathbb{Z}^+$)

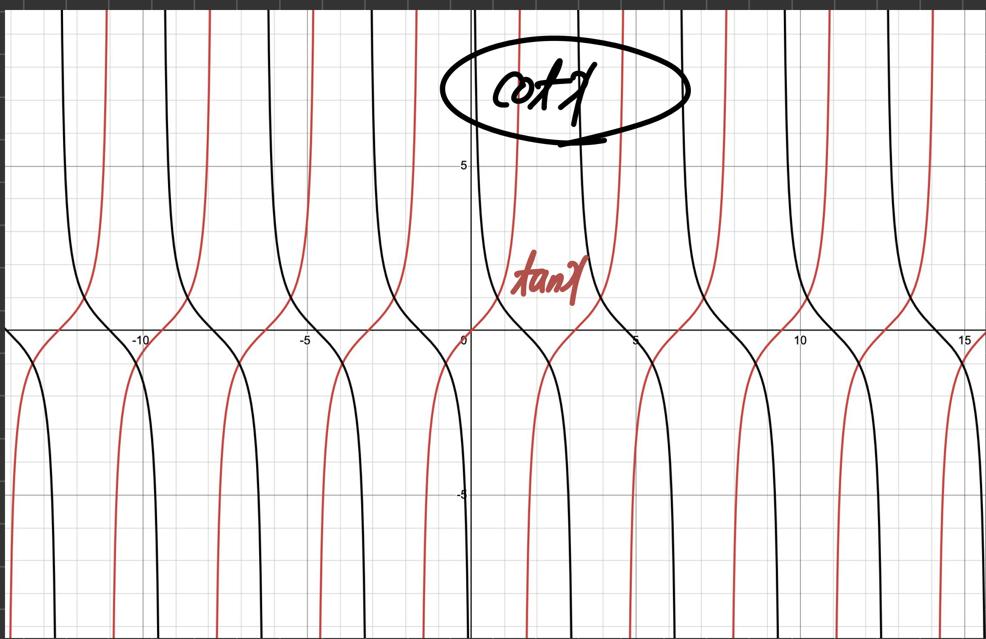
$y = x^{2k-1} \longrightarrow y = \frac{1}{x^{2k-1}}$ (s.t $k \in \mathbb{Z}^+$)



csc x
sin x
-15
-10
-5
0
5
10
15
5
-5

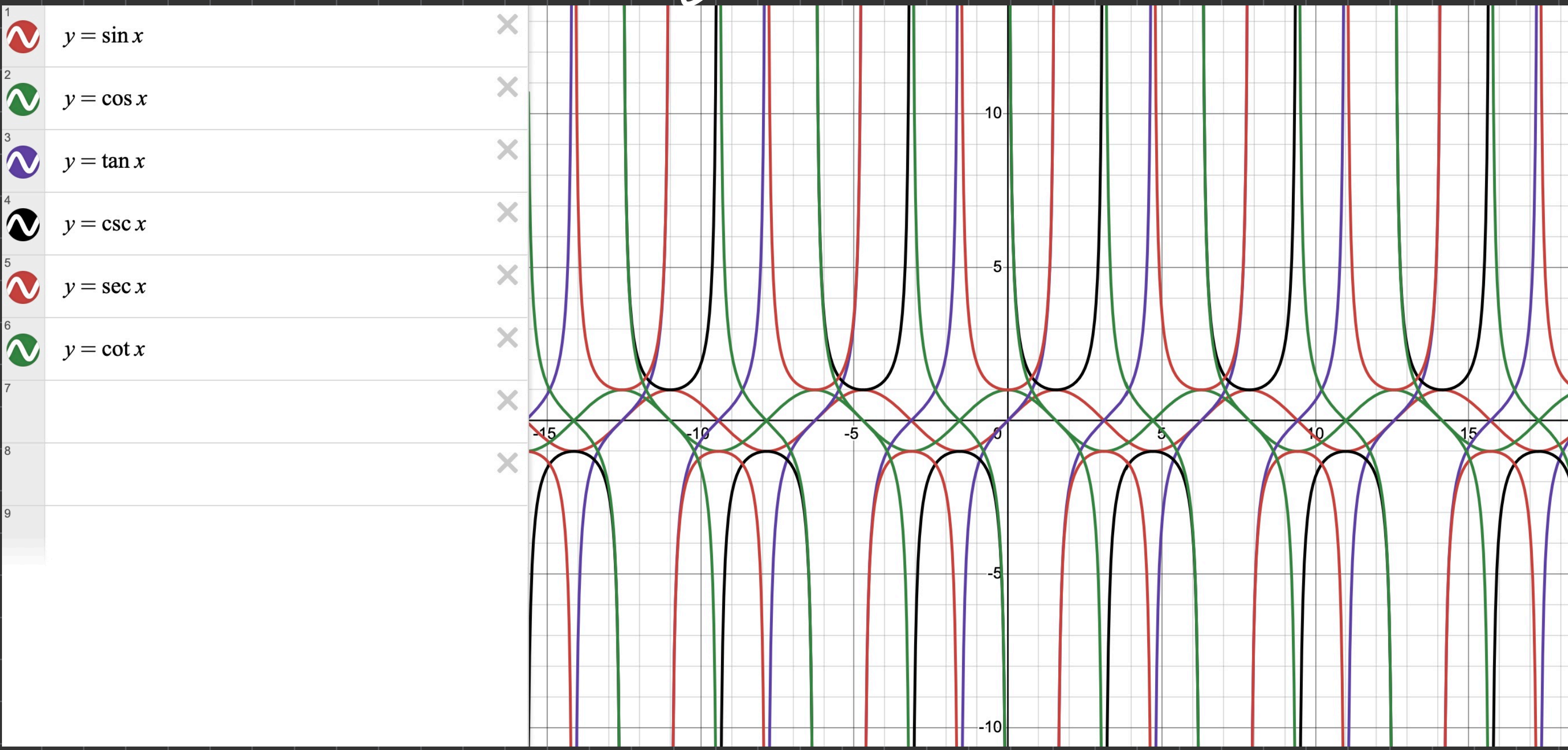


sec x
cos x
-15
-10
-5
0
5
10
15
5
-5

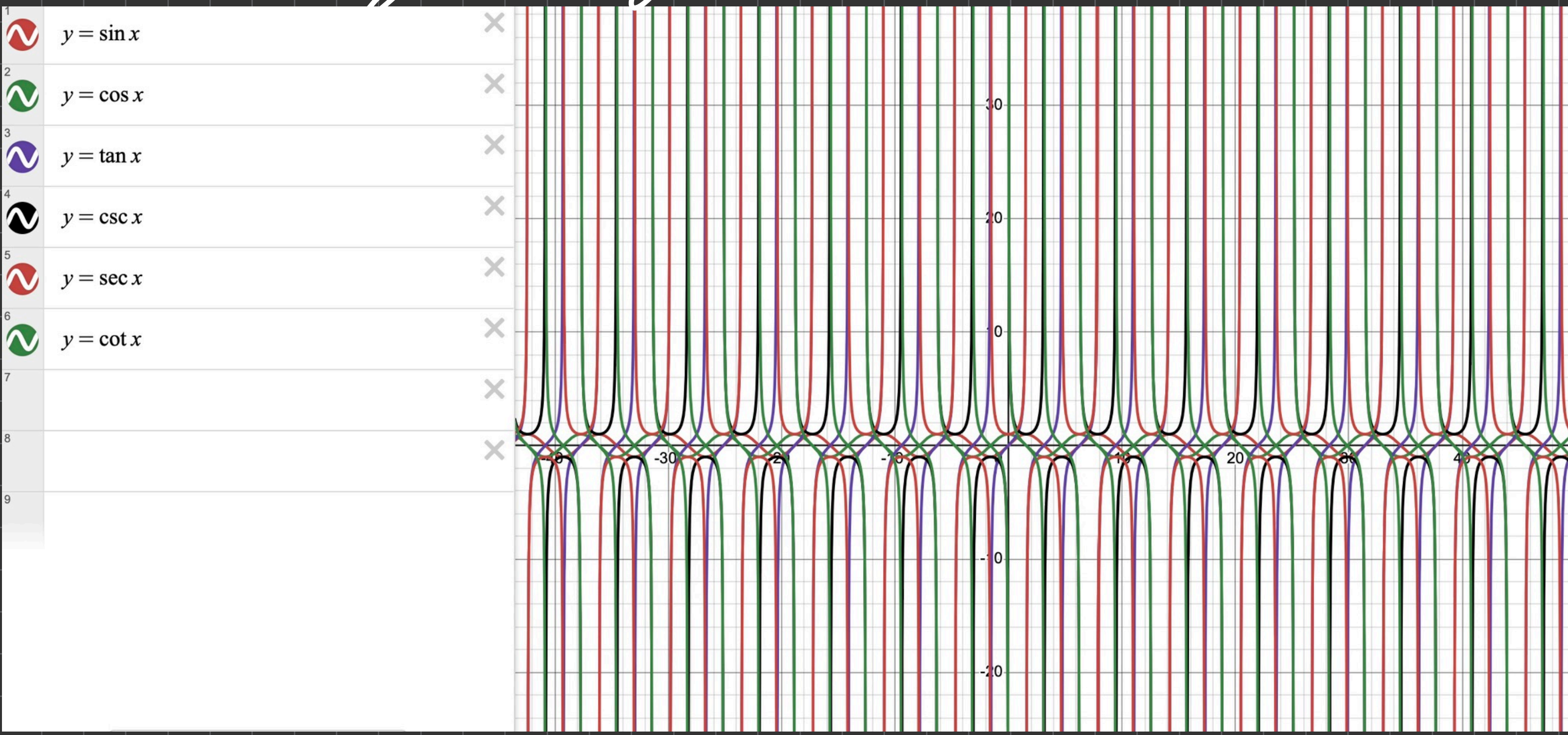


cot x
tan x
-10
-5
0
5
10
15
5
-5

Micro (trigonometry)



Macro (trigonometry)



13 Find the coefficient of x^2y^4 in the expansion of $(1+x+y^2)^7$

F

A 6

B 10

C 21

D 35

E 105

F 210

$$(x^2y^4) = (x)^2(y^2)^2(1)^{7-2-2}.$$

Coefficients: ${}_7C_2 \times {}_5C_2 \times {}_3C_3.$

$$= \frac{7\cdot 6}{2\cdot 1} \times \frac{5\cdot 4}{2\cdot 1} = 21 \times 10 = 210.$$

in trinomial (3 terms) expansion... coefficients

are $(A+B+C)^m = {}_mC_a (A)^a \times {}_{m-a}C_b (B)^b \times {}_cC_c (C)^c$

s.t $a+b+c=m$.

Generalization: $(m_1+m_2+\cdots+m_n)^s$ s.t $s = \sum_{k=1}^{n} x_k$).

$= (m_1)^{x_1} \times (m_2)^{x_2} \times \cdots \times (m_n)^{x_n} \cdot {}_sC_{x_1} \times {}_{s-x_1}C_{x_2} \times {}_{s-x_1-x_2}C_{x_3} \times \cdots \times {}_{x_n}C_{x_n}.$

* To find co-eff, check signs seperately.

9. Practice Questions.

$(A+B+C)^N$'s trinomial expansion has a term of $A^{n_a} B^{n_b} C^{n_c} \times \left({}_N C_{n_a} \times {}_{N-n_a} C_{n_b} \times {}_{N-n_a-n_b} C_{n_c} \right)$

s.t $N = n_a + n_b + n_c$

(* Caveats before computing!)

The coefficients of a trinomial expansion $(a+b+c)^n$ are generalized as multinomial coefficients given by the formula $\frac{n!}{i!\,j!\,k!}$, where i, j, and k are non-negative integers such that i + j + k = n.

The Trinomial Theorem

For any non-negative integer n, the expansion of $(a+b+c)^n$ is expressed as a finite sum over all combinations of indices i, j, k:

$$(a+b+c)^n = \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!} a^i b^j c^k$$

This notation $\frac{n!}{i!\,j!\,k!}$ is often written as the trinomial (or multinomial) coefficient $\binom{n}{i,j,k}$

9. Dive into generalisation.

(Non Maths/CS/NatSci majors can skip).

$$(m_1+m_2+m_3+\cdots+m_{n-1}+m_n)^N \left(= \left[\sum_{k=1}^{n} m_k\right]^N\right)$$

$$= (\text{Coefficient}) \times \left(m_1^{x_1}\cdot m_2^{x_2}\cdot m_3^{x_3}\cdots m_{n-1}^{x_{n-1}}\cdot m_n^{x_n}\right)$$

s.t Coefficient of this term. $\left(\sum_{k=1}^{n} x_k = N\right)$

$$\downarrow$$

$${}_N C_{x_1} \times {}_{N-x_1} C_{N-x_1-x_2} \times {}_{N-x_1-x_2} C_{N-x_1-x_2-x_3} \cdots \times {}_{N-(x_1+x_2+\cdots+x_{n-1})} C_{x_n}.$$

The generalization of expansion coefficients for a polynomial raised to the power of n is given by the Multinomial Theorem for general polynomials, and the general coefficient recurrence relations or coefficient formulas for arbitrary polynomials. روافد المعرفة للنشر العلمي +1

1. The Multinomial Theorem (Generalizing to m terms)

For a polynomial with m terms $(x_1+x_2+...+x_m)$ raised to a positive integer power n, the expansion is:

$$(x_1+x_2+...+x_m)^n = \sum_{k_1+k_2+...+k_m=n} \binom{n}{k_1,k_2,...,k_m} x_1^{k_1}x_2^{k_2}...x_m^{k_m}$$

- **Multinomial Coefficient:** The coefficient for each term is defined as:

$$\binom{n}{k_1,k_2,...,k_m} = \frac{n!}{k_1!k_2!...k_m!}$$

- **Condition:** The sum of the non-negative integer indices $k_1+k_2+...+k_m$ must always equal n. Wikipedia +1

$$(a_0x^0+a_1x^1+a_2x^2+a_3x^3+\cdots+a_{n-1}x^{n-1}+a_nx^n)^N \quad \left(\equiv\left(\sum_{k=0}^{n} a_kx^k\right)^N\right)$$

Coefficient of x^A s.t $A\in\mathbb{Z}$, $A\in[0,nN]$

in expansion of $\left(\sum_{k=0}^{n} a_kx^k\right)^N$,

(Oxbridge/Imperial Maths/Joint Maths & CS must know this truth)

$(1+x)^N$

$(1+ax)^N$

$(b+ax)^N$

$(c+bx+ax^2)^N$

$(d+cx+bx^2+ax^3)^N$

$\vdots$

Build your way up to understand generalisations.

Multinomial theorem. Let j_k be the number of brackets you take a_kx^k from. Then

$$[x^A]\left(\sum_{k=0}^{n} a_kx^k\right)^N = \sum_{\substack{j_0+j_1+\cdots+j_n=N \\ j_1+2j_2+\cdots+nj_n=A}} \frac{N!}{j_0!\,j_1!\cdots j_n!}\; a_0^{j_0}a_1^{j_1}\cdots a_n^{j_n}$$

There are two constraints. The first says you use exactly N brackets. The second says the powers add up to A.

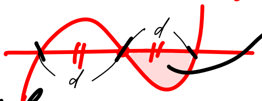
Fast computation (Miller recurrence), for $a_0\neq 0$: write $\left(\sum a_kx^k\right)^N=\sum c_mx^m$. Then

$$c_0=a_0^N, \qquad c_m=\frac{1}{m\,a_0}\sum_{k=1}^{\min(m,n)}\big(k(N+1)-m\big)\,a_k\,c_{m-k}$$

This comes from $P\cdot(P^N)'=N\,P'\cdot P^N$ by comparing coefficients of x^{m-1}. If $a_0=0$, factor out the lowest power of x first.

All $a_k=1$ (pure counting): $[x^A]=\sum_{j\geq 0}(-1)^j\binom{N}{j}\binom{A-j(n+1)+N-1}{N-1}$, by inclusion-exclusion on parts that exceed n.

☆ Likewise, if you knew formula it would be dead simple.



$S = \frac{|a|}{4}(d)^4$ if all 3 intersections' x

↗ Noir Premiere Polynomials Chpt1.

let $f(x) = ax^3 + \cdots$

14 The area enclosed between the line $y = mx$ and the curve $y = x^3$ is 6.

What is the value of m?

A 2

B 4

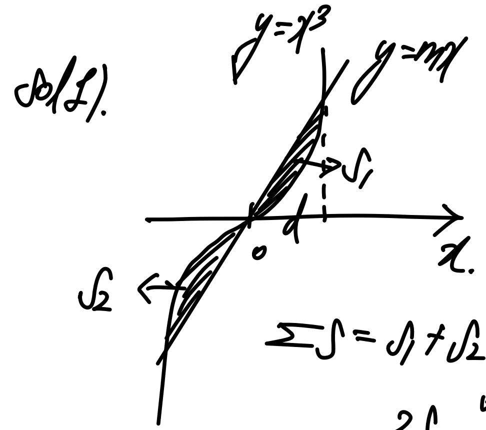
C $\sqrt{3}$

D $\sqrt{6}$

E $2\sqrt{3}$

F $2\sqrt{6}$

Sol(1).



$\Sigma S = S_1 + S_2$, $S_1 = S_2$

$2S_1$

$\frac{1}{4}|1| \cdot d^4 \times 2 = 6$

$d^4 = 12$

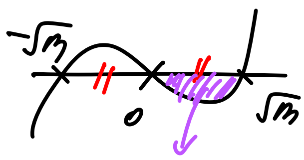
$m = d^2$

$m = \sqrt{12} = 2\sqrt{3}$.

Sol(2) let $f(x) = x^3$

$g(x) = mx$

$f(x) - g(x) = h(x) = x^3 - mx = x(x^2 - m) = x(x - \sqrt{m})(x + \sqrt{m})$.



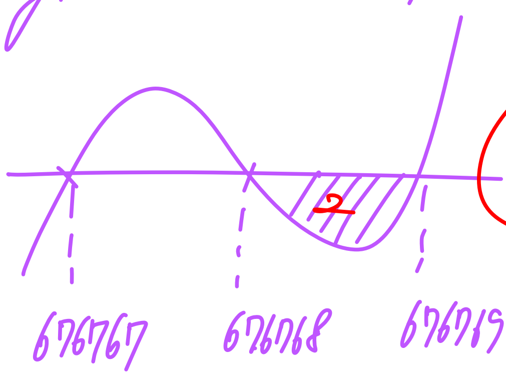
$\frac{1}{4}|1|(\sqrt{m})^4 = 3 \left(= \frac{6}{2}\right)$.

$m^2 = 12$

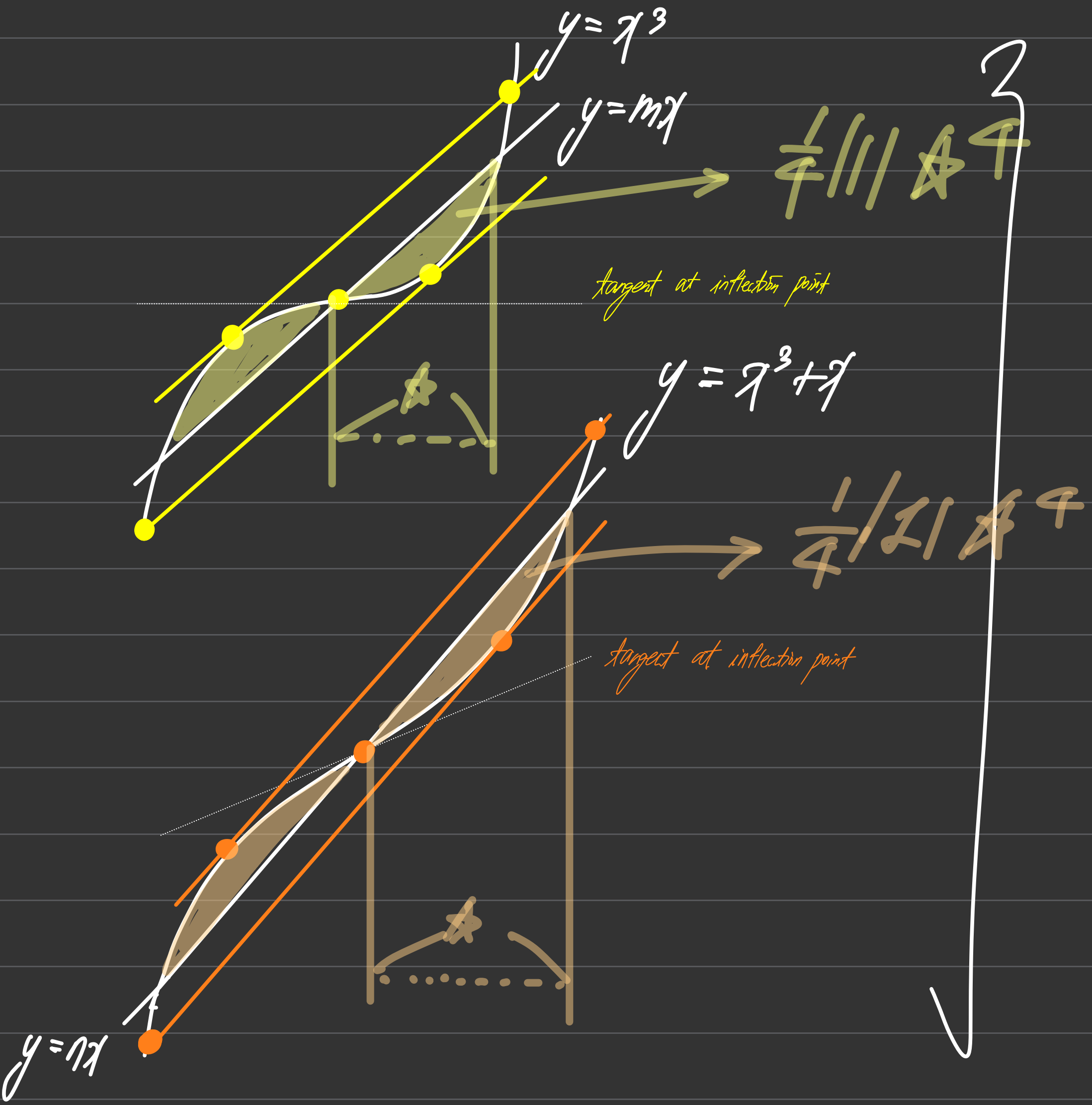
$m = 2\sqrt{3}$.

e.g.

$f(x) = 8x^3 + \sim$



The cubic function 5-star analysis does not need to be only a worm shape!



it could be the other two graphs too!

from Noir Premiere.

15 Find the positive difference between the two real values of x for which

$$(\log_2 x)^4 + 12\left(\log_2\left(\frac{1}{x}\right)\right)^2 - 2^6 = 0$$

A 4

B 16

C $\frac{15}{4}$

D $\frac{17}{4}$

E $\frac{255}{16}$

F $\frac{257}{16}$

let $\log_2 x = k$. $(k \in \mathbb{R})$

$k^4 + 12k^2 - 2^6 = 0.$

$k^4 - (2^4 - 2^2)k^2 - 2^6 = 0.$

$(k^2 - 2^4)(k^2 + 2^2) = 0.$

$= (k - 2^2)(k + 2^2)\underline{(k^2 + 2^2)} = 0.$

$\hookrightarrow k =$ Complex (imaginary).

$\therefore k = \pm 2^2$

$\log_2 x = 2^2, 2^{-2}$

$x = 4, \frac{1}{4} \rightarrow |4 - \frac{1}{4}| = \frac{15}{4}$

*If $y=mx+n$ (linear function), then $m=\tan\theta$ s.t θ is acute angle between $+x$ axis and l

9. Geometric Intuition.

16 The circle C_1 has equation $(x+2)^2+(y-1)^2=3$

The circle C_2 has equation $(x-4)^2+(y-1)^2=3$

The straight line l is a tangent to both C_1 and C_2 and has positive gradient.

(common terms) two centers have identical y-values & identical radii.

The acute angle between l and the x-axis is θ

Find the value of $\tan\theta$

A $\frac{1}{2}$

B 2

C $\frac{\sqrt{2}}{2}$

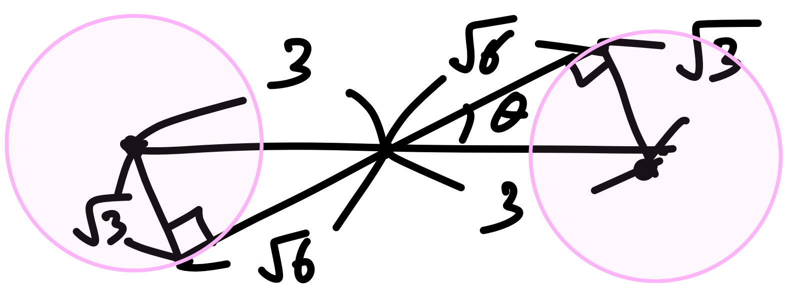
D $\sqrt{2}$

E $\frac{\sqrt{6}}{2}$

F $\frac{\sqrt{6}}{3}$

G $\frac{\sqrt{3}}{3}$

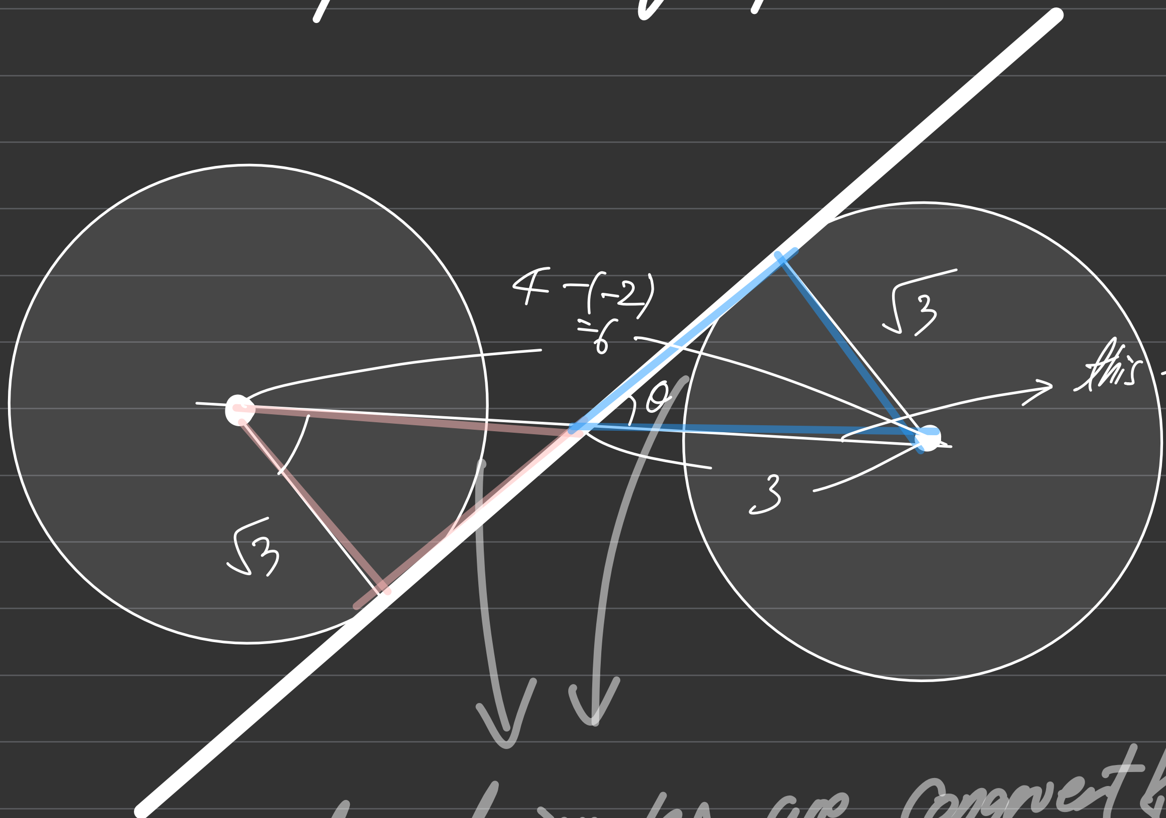
H $\sqrt{3}$



$$\tan\theta = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

* Critical thought: Same Radius ⇒ two tangential triangles are congruent.

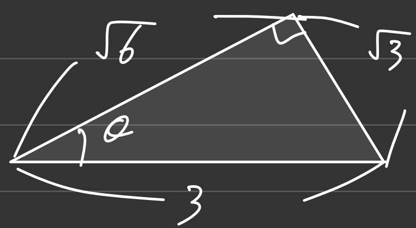
Don't overperfect graph.



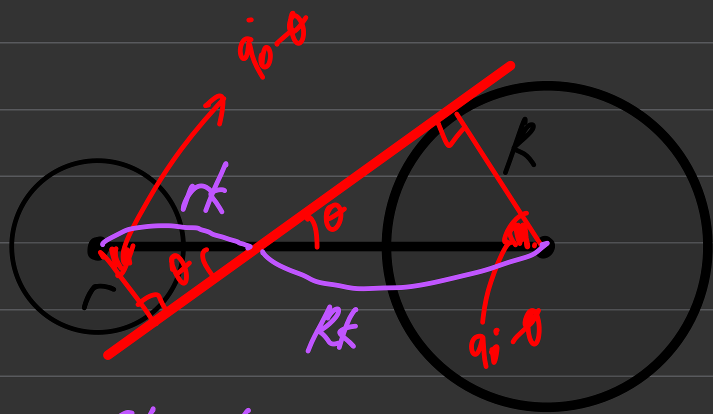
this line in this case is parallel to x-axis

two triangles are congruent!

So simply find $\tan\theta$ from



$$\tan\theta = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$



$(R+r)k = d.$

☆ Look at the extreme boundary case scenarios & start working your way towards in between.

17 Find the complete set of values of m in terms of c such that the graphs of $y = mx + c$ and $y = \sqrt{x}$ have two points of intersection.

A $0 < m < \frac{1}{4c}$

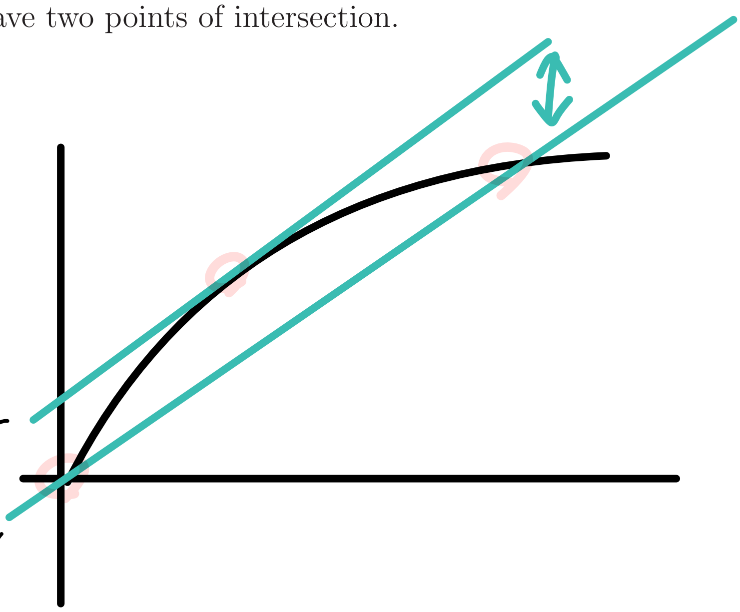
B $0 < m < 4c^2$

C $m > \frac{1}{4c}$

D $m < \frac{1}{4c}$

E $m > 4c^2$

F $m < 4c^2$



$\sqrt{x} = mx + c$ & $x > 0$.

$x = m^2x^2 + 2mcx + c^2$

$m^2x^2 + (2mc-1)x + c^2$

$D = (2mc-1)^2 - 4m^2c^2$

$= 4m^2c^2 - 4mc + 1 - 4m^2c^2$

$= 1 - 4mc = 0.$

$\therefore m = \frac{1}{4c}$

when $c = 0.$ & $m > 0.$

☆ Get used to case division. splitting into piecewise functions.

18 Find the number of solutions and the sum of the solutions of the equation

$$1 - 2\cos^2 x = |\cos x|$$

where $0 \leq x \leq 180°$

A Number of solutions $= 2$ Sum of solutions $= 180°$

B Number of solutions $= 2$ Sum of solutions $= 240°$

C Number of solutions $= 3$ Sum of solutions $= 180°$

D Number of solutions $= 3$ Sum of solutions $= 360°$

E Number of solutions $= 4$ Sum of solutions $= 240°$

F Number of solutions $= 4$ Sum of solutions $= 360°$

(i) $\cos x \geq 0$.

$1 - 2\cos^2 x = \cos x$.

$2\cos^2 x + \cos x - 1 = 0$.

$2 \quad -1$

$1 \quad 1$

$(2\cos x - 1)(\cos x + 1) = 0$.

$\cos x = \frac{1}{2}$.

(ii) $\cos x \leq 0$

$1 - 2\cos^2 x = -\cos x$

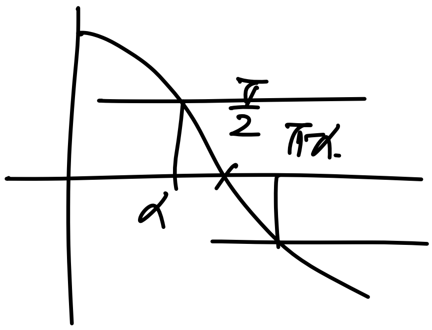
$2\cos^2 x - \cos x - 1 = 0$.

$2 \quad 1$

$1 \quad -1$

$(2\cos x + 1)(\cos x - 1) = 0$.

$\cos x = -\frac{1}{2}$.



☆ There was no need to sketch the graphs.

Common term) $-52x-52$

→ Common term expressed... So group them.

☆ Critical thought.

19 Find the lowest positive integer for which $x^2 - 52x - 52$ is positive.

A 26

B 27

C 51

D 52

E 53

F 54

$\min(x)$ s.t $x^2-52x-52>0.$

$x^2>52(x+1)$

at $x=51$:

$51^2>52\times52$. (X)

at $x=52$.

$52^2>52\times53$ (X)

$\therefore$ at $x=53$

$53^2>\underbrace{52\times54}$

$\hookrightarrow(53-1)(53+1)=53^2-1.$

☆ Repeated Roots could be triple roots!

→ Two distinct roots ⇒ one roots is repeated.

9. Caveats.

20 For how many values of a is the equation

$$(x-a)(x^2-x+a)=0$$

satisfied by exactly two distinct values of x?

A 0

B 1

C 2

D 3

E 4

F more than 4

(i) $(x-a) \mid (x^2-x+a)$ and $x=a$ is not triple root.

$a^2-a+a=0 \to a=0.$ $\quad x(x^2-x).$

(ii) $x-a \nmid (x^2-x+a)$ and (x^2-x+a) has repeated root.

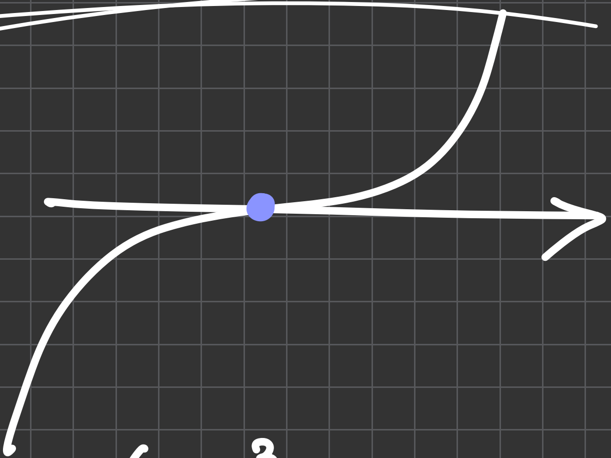
$\to D=0,\ 1-4a=0$

$a=\frac{1}{4}.$

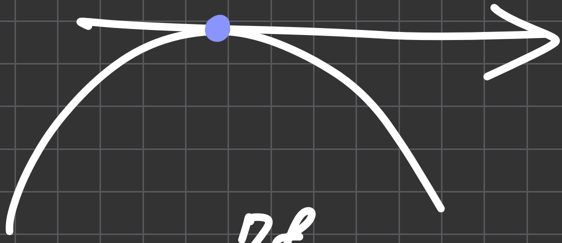
$(x-\frac{1}{4}) \nmid (x-\frac{1}{2})^2.$

$\therefore a=\frac{1}{4}$ is valid.

9. Caveats

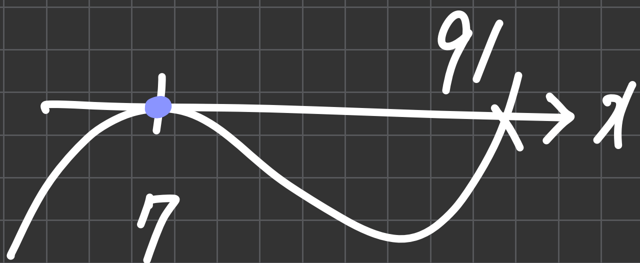


$y=(x-1)^3$

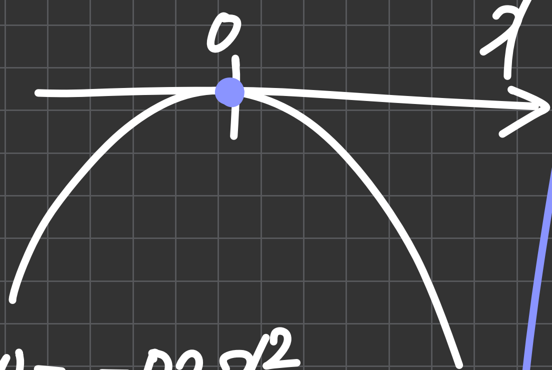


$y=-(x-7)^{78}$

* Both are still verbally expressed as "Repeated Roots"!



$f(x)=9(x-7)^2(x-91)$



$f(x)=-72x^2$

Don't react as "repeated" only when it shows up as twice.

[In technical terms, a repeated root can show up any number of times greater than one].

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TMUA 9.
Jamie Baek.

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