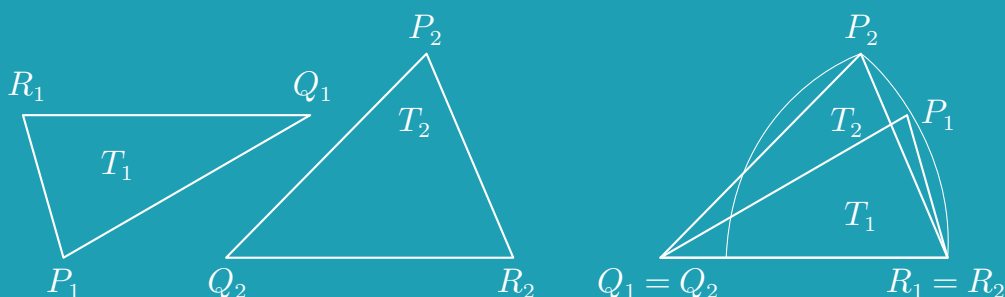


THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

2001–2016**Problems, Solutions, and Commentary**

Kiran S. Kedlaya

Daniel M. Kane

Jonathan M. Kane

Evan M. O'Dorney

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**Problems, Solutions, and
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Dedicated to the Putnam contestants

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Introduction

This book is the fourth collection of William Lowell Putnam Mathematical Competition problems and solutions, following in the footsteps of [PutnamI] (1938–1964), [PutnamII] (1965–1984), and [PutnamIII] (1985–2000). We have largely followed the structure of the third collection, by including multiple solutions and commentary putting the problems in a broader mathematical context.

The Putnam Competition. The William Lowell Putnam Mathematical Competition (a/k/a “the Putnam”) is the preeminent mathematics competition for undergraduate college students in the United States and Canada, and one of the oldest and most prominent such competitions in the world. The first competition was held in 1938; this book covers the 62nd–77th competitions, held in the years 2001–2016. Since 1962, the competition has been held on the first Saturday in December, consisting of two three-hour sessions of six problems each (A1–A6 in the morning, B1–B6 in the afternoon). For more information about the history of the Putnam Competition, see the articles of Garrett Birkhoff and L. E. Bush in [PutnamI], and also [Arney].

The competition is open to regularly enrolled undergraduates in the U.S. and Canada who have not yet received a college degree; the 2016 competition saw more than 4000 contestants from over 500 institutions. No individual may participate in the competition more than four times.

Since its origin, the competition has been organized by the Mathematical Association of America and funded by the William Lowell Putnam Prize Fund for the Promotion of Scholarship. From 1975 to 2017, the competition was administered by Leonard Klosinski (director) and Gerald Alexanderson (associate director) at Santa Clara University; from 1985 to 2017, Loren Larson also served as an associate director. As of the 2017 competition (not included in this volume), the competition is being administered directly by the MAA, with Daniel Ullman as director and Mark Krusemeyer as associate director.

The problems. The competition is meant to cover the standard undergraduate mathematics curriculum; at present, this is generally taken to mean calculus,

linear algebra, differential equations, real analysis, abstract algebra, number theory, probability, and combinatorics. (In some cases, in addition to the intended solution, a problem may admit an alternate solution using more advanced mathematics.) However, problems often involve nonroutine applications of these tools, to the extent that a standard mathematical education is not necessarily the best preparation for the competition; experience with high-school competitions such as the USA Mathematical Olympiad (USAMO) and the International Mathematical Olympiad (IMO) is somewhat more transferable, although those competitions generally exclude calculus and some other advanced topics.

The problems are set by the Questions Committee, typically consisting of three mathematicians appointed by the MAA for staggered three-year terms, with the member whose term is closest to expiration serving as Chair; the committee is assisted in this effort by the associate director. For each competition included in this book, we have included the membership of the Questions Committee together with the problem statements. (The Questions Committee does not issue attributions for individual problems.)

Results of the competition. While each contestant takes the exam individually, the competition includes both individual and team aspects. On the individual side, top-scoring contestants are recognized in the following tiers. (The size of each group is only indicative; it is subject to small fluctuations due to tied scores.)

- The top 5 contestants, designated as Putnam Fellows. These receive a cash prize; one of the Putnam Fellows is also awarded the William Lowell Putnam Prize Scholarship at Harvard.
- The next 10 contestants. These receive a cash prize.
- The next 10 contestants. These receive a cash prize.
- The remaining contestants¹ among the top 100 finishers. These are designated Honorable Mentions.
- The remaining contestants among the top 200 finishers.
- The remaining contestants among the top 500 finishers.

Beginning in 1992, the Elizabeth Lowell Putnam Prize has been awarded in some years to a woman whose performance has been deemed particularly meritorious. The winner receives a cash prize.

On the team side, each participating college or university with at least three participants is awarded a team score equal to the sum of the top three scores from

¹This is the definition in use at the time of writing. Previously, in addition to the top 100, there was a smaller Honorable Mention tier; the official competition reports reflect this distinction.

that institution.² The top 5 teams are recognized as such (in order), with cash prizes awarded both to the mathematics departments of the institutions and to the team members. The next 5 teams are designated Honorable Mentions.

An Announcement of Winners, including all of the designations listed above, is released by the MAA several months after the competition. An official report of each competition is published in the *American Mathematical Monthly* in the September or October issue of the following year; this report includes the problems, a complete set of solutions, individual results (through Honorable Mention, plus the Elizabeth Lowell Putnam Prize), and team results. It also includes, for each problem, a breakdown of how many of the top 200 contestants obtained each possible score on each problem; note that while problems are graded out of 10 points each, generally only the scores 0, 1, 2, 8, 9, 10 are used.³

It should be emphasized that the competition is intended not merely to identify winners, but also to provide a challenge to all of the contestants. As scores can be quite low (the median score is commonly 0), even solving a single problem is a notable result!

Structure of this book. The first section contains the problems, as they originally appeared in the competition. Next is a section containing a brief hint for each problem. The hints may often be more mystifying than enlightening. Nonetheless, we hope that they encourage readers to spend more time wrestling with a problem before turning to the solutions section.

The heart of this book is in the solutions. (Beware that in certain instances, our standard notation disagrees with the problem statement; these are described at the front of the solutions section.) For each problem, we include one or more solutions; multiple solutions generally illustrate distinct ideas, although the later solutions are compressed when they partly repeat an earlier solution. We also include commentary in the form of remarks on the various solutions. In addition, with each problem's solution we give the "score vector" consisting of the number of participants from among the top 200 finishers achieving each possible score on the problem; for ease of reference, we also indicate the percentage of the top 200 finishers achieving a score of 8–10, which we call the *success rate*. (See the first page of the solutions section for a precise definition of the score vector.)

After the solutions comes a tabulation of competition results for the years 2001–2016, including tables of various score cutoffs, including the individual tiers of recognition listed above; a list of Putnam Fellows; the top 5 schools in the team competition; and a detailed analysis of the results by Joe Gallian. More detailed

²This is the rule in use at the time of writing. During the years covered in this volume, a different system was used; see the team results section for details.

³This restriction was phased out as of the 2018 competition, but scores in the 3–7 range are likely to remain rare in the future.

year-by-year results can be found at the Putnam Archive (see below) and in the official competition reports in the *American Mathematical Monthly* (see above).

In addition to a standard keyword index, we have also included a topic index for the problems. This constitutes a rough (and necessarily subjective) identification of which topics from the undergraduate mathematics curriculum pertain to each problem treated in this book.

Acknowledgments. Our primary source for problem statements and competition results, and a secondary source for problem solutions, is the official competition reports (see above). We thank the Mathematical Association of America and the William Lowell Putnam Mathematical Competition for the use of the problems and results from the various Competitions.

Our primary source for the solutions published in this volume has been the online Putnam Archive, maintained by one of us (Kedlaya) at

<http://kskedlaya.org/putnam-archive>.

The Putnam Archive includes solutions compiled by Kedlaya together with Manjul Bhargava and Lenhard Ng; we thank Bhargava and Ng for their permission to use that material as the basis for this book. We are also grateful to the many individuals who have shared ideas that are incorporated into the solutions, including the Art of Problem Solving (AoPS) online community at <http://artofproblemsolving.com>; we have attributed these contributions to the best of our ability. Special recognition is due to Kent Merryfield, who built the AoPS Putnam forum into a definitive presence, and maintained it as such until his untimely passing in November 2018; his efforts will be dearly missed.

We thank Joe Gallian for helping to assemble the competition results and with making his analysis available to us for inclusion. We also wish to acknowledge Joe's continuing role as unofficial William Lowell Putnam Competition historian as well as his mentoring of undergraduate students, from which all four of us have benefited directly.

We thank Steve Kennedy for his continued support of this project, and the AMS/MAA Problem Books editorial board for their detailed feedback on a draft of this book.

We thank the Mathematical Association of America for continuing to organize the competition; the past directors Gerald Alexanderson, Leonard Klosinski, and Loren Larson for their extended service to the mathematical community; the current director Daniel Ullman and associate director Mark Krusemeyer for carrying this work forward; numerous unnamed volunteers who have participated in the grading of the competition; and the Putnam family for its continued support of the competition over so many years.

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Topic Index

In the following index, we identify problems referring to the following topics.

- Algebra
- Induction
- Polynomials
- Inequalities
- Geometry
- Trigonometry
- Limits
- Differentiation
- Integration
- Series/Products
- Differential Equations
- Real Analysis
- Complex Analysis
- Combinatorics
- Enumerative Combinatorics
- Probability
- Pigeonhole
- Linear Algebra
- Determinants
- Number Theory

- Abstract Algebra
- Group Theory
- Finite Fields
- Functional Equations
- Generating Functions
- Recurrence Relations
- Game Theory
- Optimization

An extended version of this index can be found online at the Putnam Archive.

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