TMUA9. Model Answer.

Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION **D513/12**

PAPER 2

Wednesday 31 October 2018 Time: 75 minutes

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the second of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

Any rough work should be done on this question paper. No extra paper is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

Calculators and dictionaries must **NOT** be used.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV1

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TMUA9.

Jamie Baek

(백선우)

1 The function f is given, for $x > 0$, by

$$f(x) = \frac{x^3 - 4x}{2\sqrt{x}}$$

Find the value of $f'(4)$.

A 3

B 9

C 9.5

D 12

E 39.5

F 88

$f(x) = \frac{1}{2}x^{-\frac{1}{2}}(x^3 - 4x).$

$= \frac{1}{2}(x^{\frac{5}{2}} - 4x^{\frac{1}{2}}).$

$f'(x) = \frac{1}{2}(\frac{5}{2}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}).$

Don't need to simplify further as the goal is $f'(x)$ only!

$f'(4) = \frac{1}{2}(\frac{5}{2} \times 8 - 2 \times \frac{1}{2})$

$= \frac{1}{2}(19)$

$^{12}C_3 = {}^{12}C_9$

2 Find the value of the constant term in the expansion of

$$\left(x^6 - \frac{1}{x^2}\right)^{12}$$

A -495

B -220

C -66

D 66

E 220

F 495

$6 \times 3 = 2 \times 9.$

$\therefore (x^6)^3 \times \left(-\frac{1}{x^2}\right)^9 \rightarrow$ Constant.

$^{12}C_3 \cdot (x^6)^3 \cdot \left(-\frac{1}{x^2}\right)^9$

$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220.$

$\therefore (-1)^9 \cdot 220 \rightarrow 220.$

2 Find the value of the constant term in the expansion of

$$\left(x^6 - \frac{1}{x^2}\right)^{12}$$

A -495

B -220

C -66

D 66

E 220

F 495

method 2)

$$(x^6)^A \left(-\frac{1}{x^2}\right)^B$$

$$A+B=12.$$

$$x^{6A} \cdot (-x^{-2})^B$$

$$= x^{6A-2B} \cdot (-1)^B$$

$$6A-2B=0$$

$$B=3A.$$

$$4A=12.$$

9. Visual Representation of this.

3 Consider the following statement:

> A car journey consists of two parts. In the first part, the average speed is u km/h. In the second part, the average speed is v km/h. Hence the average speed for the whole journey is $\frac{1}{2}(u+v)$ km/h.

(1) Condition true (2) Statement is not true

Which of the following examples of car journeys provide(s) a **counterexample** to the statement?

I In the first part of the journey, the car travels at a constant speed of 50 km/h for 100 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

V_1 & V_2 differ
d 不 constant.

II In the first part of the journey, the car travels at a constant speed of 50 km/h for one hour. In the second part of the journey, the car travels at a constant speed of 40 km/h for one hour.

V_1 & V_2 different
t 不 constant

III In the first part of the journey, the car travels at a constant speed of 50 km/h for 80 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

X.

d_1 & d_2 different

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

$$\frac{\Sigma \text{ total Distance.}}{\Sigma \text{ total time}}$$

$$\frac{80\text{km} + 100\text{km}}{\left(\frac{80\text{km}}{50\text{km/h}} + \frac{100\text{km}}{40\text{km/h}}\right)}$$

4 The non-zero real number c is such that the equation $\cos x = c$ has two solutions for $0 < x < \frac{3}{2}\pi$.

How many solutions of the equation $\cos^2 2x = c^2$ are there in the range $0 < x < \frac{3}{2}\pi$?

A 2

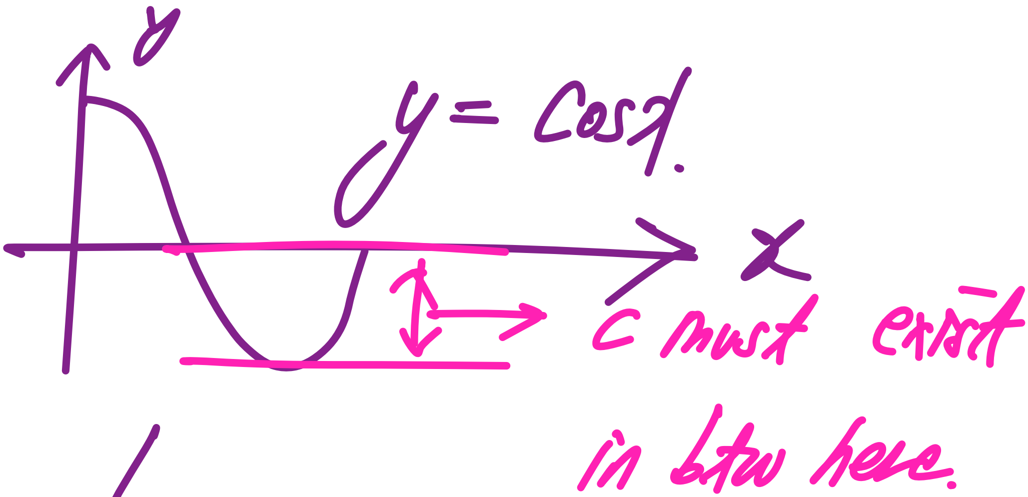
B 3

C 4

D 6

E 7

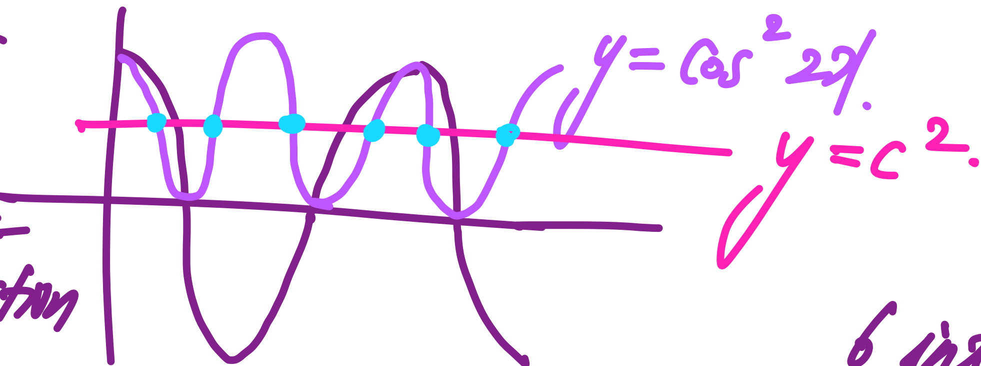
F 8



$\therefore -1 < c < 0.$

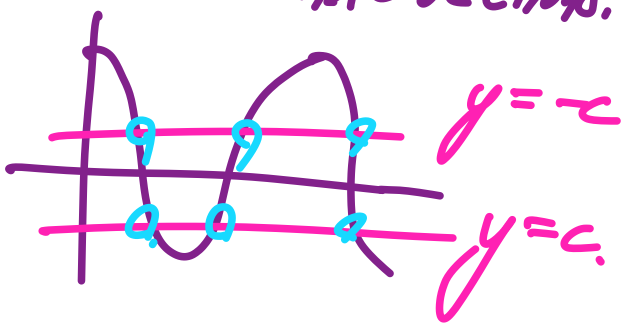
sol(1) $y = \cos^2 2x$ vs $y = c^2$ $(0 < c^2 < 1)$

*6 points of inter-section



6 intersections.

sol(2) $y = \cos 2x$ and $\begin{cases} y = c \\ y = -c \end{cases}$



4 The non-zero real number c is such that the equation $\cos x = c$ has two solutions for $0 < x < \frac{3}{2}\pi$.

How many solutions of the equation $\cos^2 2x = c^2$ are there in the range $0 < x < \frac{3}{2}\pi$?

$f(x)$

① increasing/decreasing $f'(x)$

② concave up/down $f''(x)$

piecewise function.

A 2

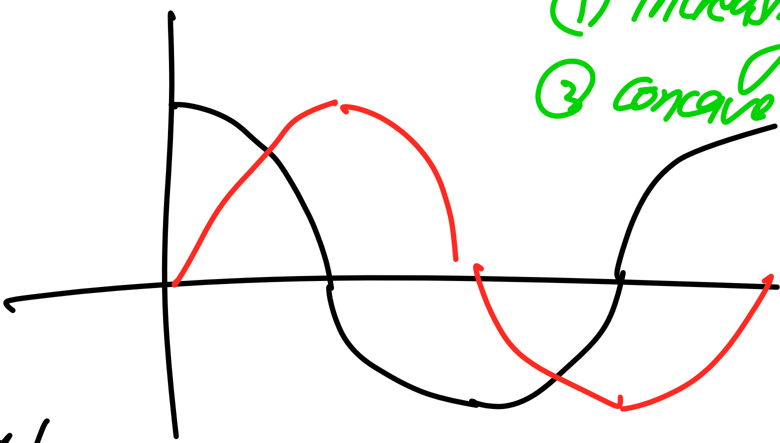
B 3

C 4

D 6

E 7

F 8



$|\cos x|$

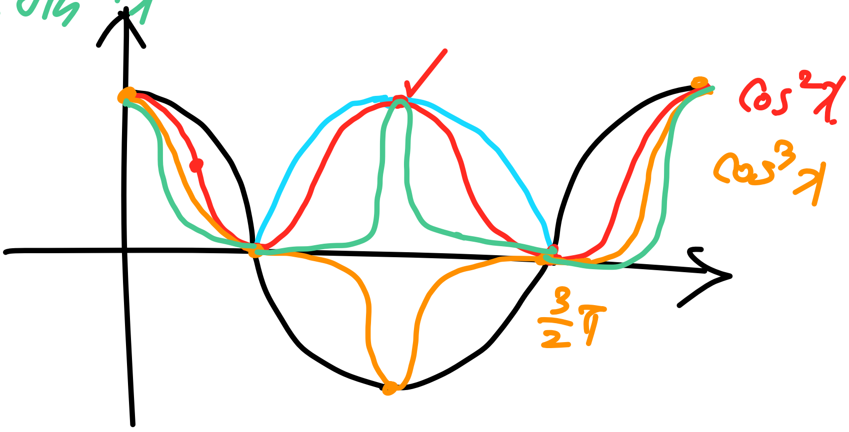
$\cos^2 x \to \sin^2 x$

$\to \sin^3 x$

$\cos^3 x \to \sin^4 x$

$\cos^4 x.$

$\Downarrow$



$1^3 = 1$

$(-1)^3 = -1$

Check Noir Premiere for geometry of squares.

5 The two diagonals of the quadrilateral Q are perpendicular.

9. Midpoints of quadrilaterals of all types.

Consider the following statements:

I One of the diagonals of Q is a line of symmetry of Q.

II The midpoints of the sides of Q are the vertices of a square.

Which of these statements is/are **necessarily** true for the quadrilateral Q?

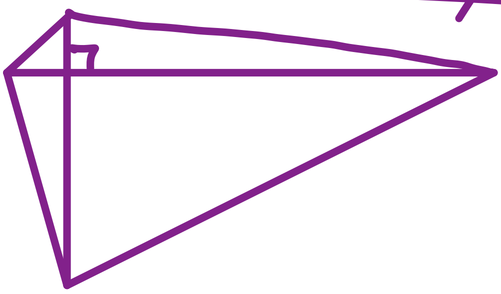
A neither of them

B I only

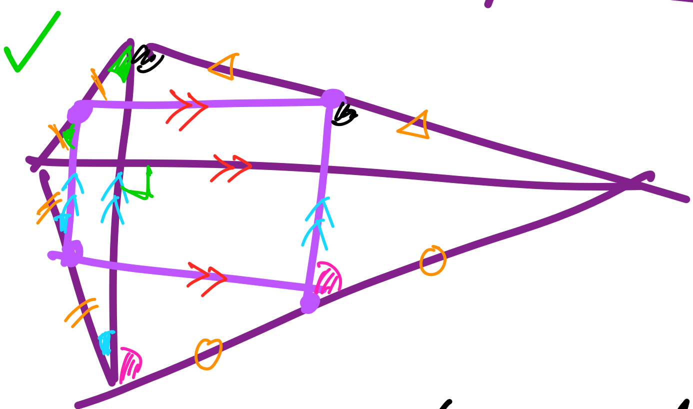
C II only

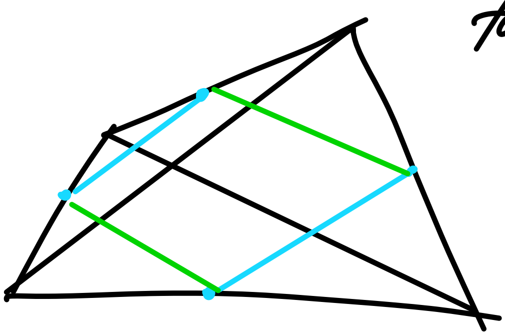
D I and II

I) Counterexample:

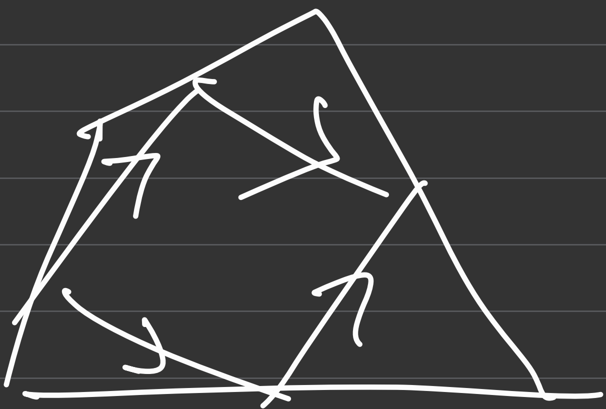


II) Canterexample:

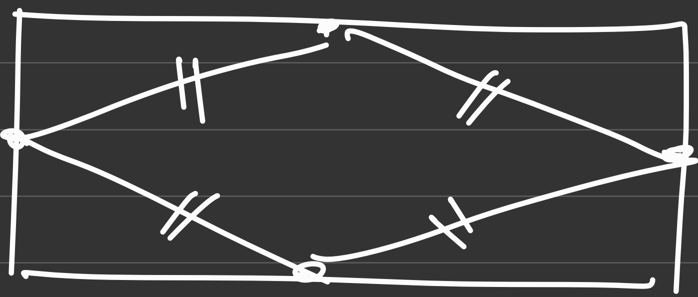




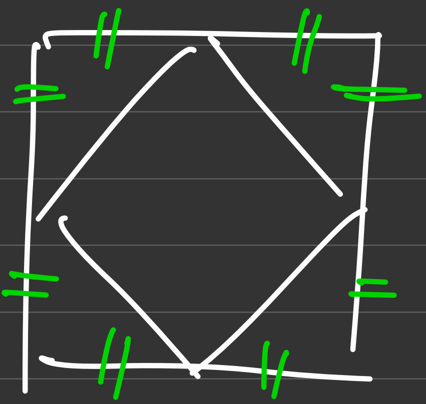
taking midpoints of a quadrilateral gives p →.



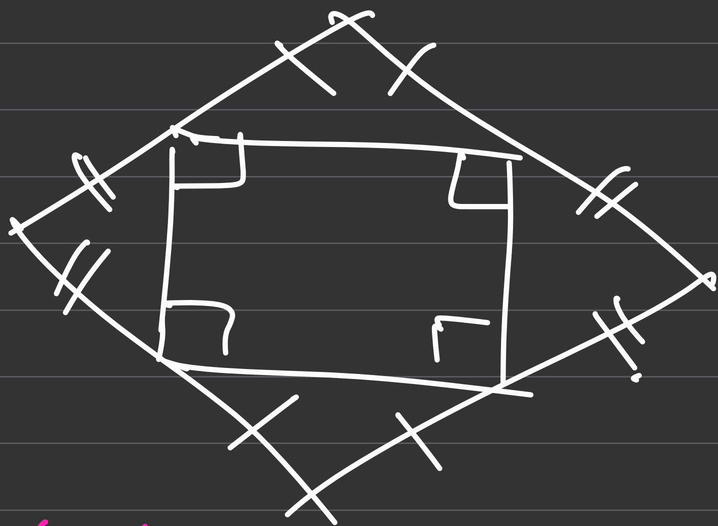
quadrilateral → parallelogram



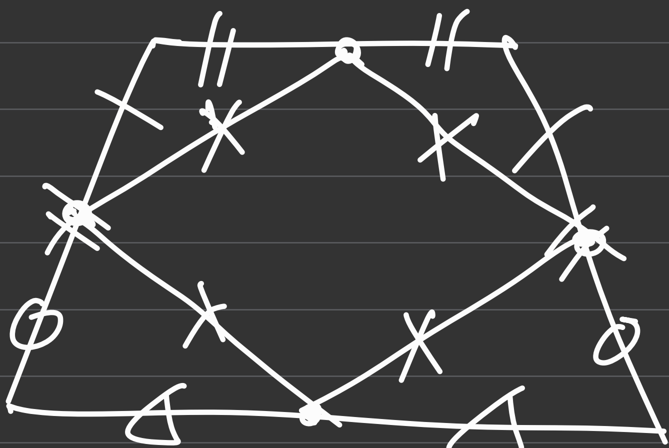
Rectangle → Rhombus



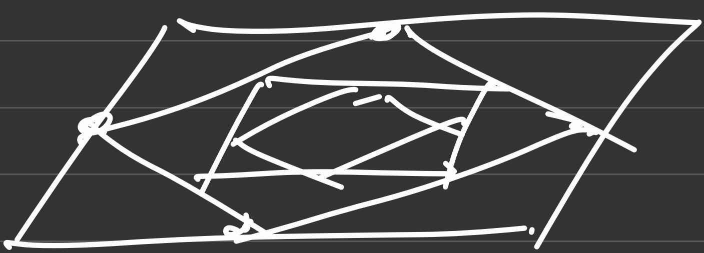
Square → Square



rhombus → rectangle



isosceles trapezoid → Rhombus



parallelogram → parallelogram

↗. Check Noir Premiere's properties of cubics.

"Every cubic function must have at least one x-intercept.

6 Which one of the following functions provides a **counterexample** to the statement:

$$\textbf{if } f'(x) > 0 \text{ for all real } x, \textbf{ then } f(x) > 0 \text{ for all real } x.$$

A $f(x) = x^2 + 1$

B $f(x) = x^2 - 1$

C $f(x) = x^3 + x + 1$

D $f(x) = 1 - x$

E $f(x) = 2^x$

$\{p \to q\}$ is counter.

$\Rightarrow$
- p is true. (so condition is true)
- q is false. (Output is false.)

C. $f'(x) = 3x^2+1 > 0.$

① so $f'(x) > 0$

(Condition is true)

② but $f(x)$ is a cubic, and $f(x)$ looks like

shape.

↗. Check 「Noir」 for the 3 types of curves

9. Generalisation of identifying common terms between two sequences.

Sequence 1 is an arithmetic progression with first term 11 and common difference 3.

Sequence 2 is an arithmetic progression with first term 2 and common difference 5.

Some numbers that appear in Sequence 1 also appear in Sequence 2. Let N be the 20th such number.

What is the remainder when N is divided by 7?

A 0

B 1

C 2

D 3

E 4

F 5

G 6

a_3 ✓

$a_n = 11 \quad 14 \quad \underline{17} \quad 20 \quad 23 \quad 26 \quad 29 \quad 32$

$b_n = 2 \quad 7 \quad 12 \quad \underline{17} \quad 22 \quad 27 \quad 32 \quad 37 \ldots$

$a_3 = b_4$ $\qquad b_4$

$b_n - a_n$

$= (5n-3)$

$-(3n+8)$

$= \underline{2n}-11$

Hence, common terms:

$U_n: \quad 17 \xrightarrow{} 32. \xrightarrow{+15} 47. \ldots$

$+15 (= 3 \times 5)$

$= 17 + 15(n-1)$

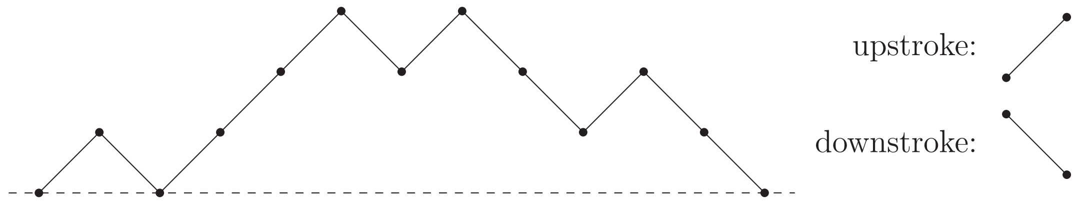
$N = U_{20} = 17 + 15 \times 19$

$\equiv 3 + 1 \times 5 \equiv 1 \pmod 7$

do (2). if you don't know number theory

Just calculate it.

8 The diagram shows an example of a *mountain profile*.



This consists of *upstrokes* which go upwards from left to right, and *downstrokes* which go downwards from left to right. The example shown has six upstrokes and six downstrokes. The horizontal line at the bottom is known as *sea level*.

A *mountain profile of order* n consists of n upstrokes and n downstrokes, with the condition that the profile begins and ends at sea level and **never** goes **below** sea level (although it might reach sea level at any point). So the example shown is a mountain profile of order 6.

Mountain profiles can be coded by using U to indicate an upstroke and D to indicate a downstroke. The example shown has the code UDUUUDUDDDUDD. A sequence of U's and D's obtained from a mountain profile in this way is known as a *valid code*.

Which of the following statements is/are true?

Obvious conterexample. ✓

~~I~~ If a valid code is written in reverse order, the result is always a valid code.

~~II~~ If each U in a valid code is replaced by D and each D by U, the result is always a valid code.

Obviously not it's reflected downwards

III If U is added at the beginning of a valid code and D is added at the end of the code, the result is always a valid code.

push upwards translations

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

* You can't cross out stuff without verification that it is not $\frac{1}{2}$.

9 Consider the following attempt to solve the equation $4x\sqrt{2x-1} = 10x - 5$:

$$4x\sqrt{2x-1} = 10x - 5$$

$\Big\downarrow$ (I)

$$4x\sqrt{2x-1} = 5(2x-1)$$

$\Big\downarrow$ (II)

$$16x^2(2x-1) = 25(2x-1)^2$$

$\Big\downarrow$ (III)

$$16x^2 = 25(2x-1)$$

$\Big\downarrow$ (IV)

$$16x^2 - 50x + 25 = 0$$

$\Big\downarrow$ (V)

$$(8x-5)(2x-5) = 0$$

$\Big\downarrow$ (VI)

The solutions of the original equation are $x = \dfrac{5}{8}$ and $x = \dfrac{5}{2}$.

Which one of the following is true?

A The solution is correct.

B Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (II).

C Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (III).

D Only one of $x = \frac{5}{8}$ and $x = \frac{5}{2}$ is correct and the error arises as a result of step (IV).

E There is another value of x that satisfies the original equation and the error arises as a result of step (II).

F There is another value of x that satisfies the original equation and the error arises as a result of step (III). $x = \frac{1}{2}$.

G There is another value of x that satisfies the original equation and the error arises as a result of step (IV).

f(☆) = f(△) when ☆+△ = 2a s.t a is constant, that means

10 The function $\mathrm{f}(x)$ is defined for all real numbers.

Consider the following three conditions, where a is a real constant:

I $\mathrm{f}(a - x) = \mathrm{f}(a + x)$ for all real x.

II $\mathrm{f}(2a - x) = \mathrm{f}(x)$ for all real x.

III $\mathrm{f}(a - x) = \mathrm{f}(x)$ for all real x.

Which of these conditions is/are **necessary and sufficient** for the graph of $y = \mathrm{f}(x)$ to have reflection symmetry in the line $x = a$?

	Condition I is necessary and sufficient	Condition II is necessary and sufficient	Condition III is necessary and sufficient
A	yes	yes	yes
B	yes	yes	no
C	yes	no	yes
D	yes	no	no
E	no	yes	yes
F	no	yes	no
G	no	no	yes
H	no	no	no

☆ necessary and sufficient

≡ if and only if ≡ iff = constant.

$a = 2$

11 Consider the equation $2^x = mx + c$, where m and c are real constants.

Which of the following statements is/are true?

A

- I The equation has a negative real solution **only if** $c > 1$.
- II The equation has two distinct real solutions **if** $c > 1$.
- III The equation has two distinct positive real solutions **if and only if** $c \leq 1$.

A none of them

B I only

C II only

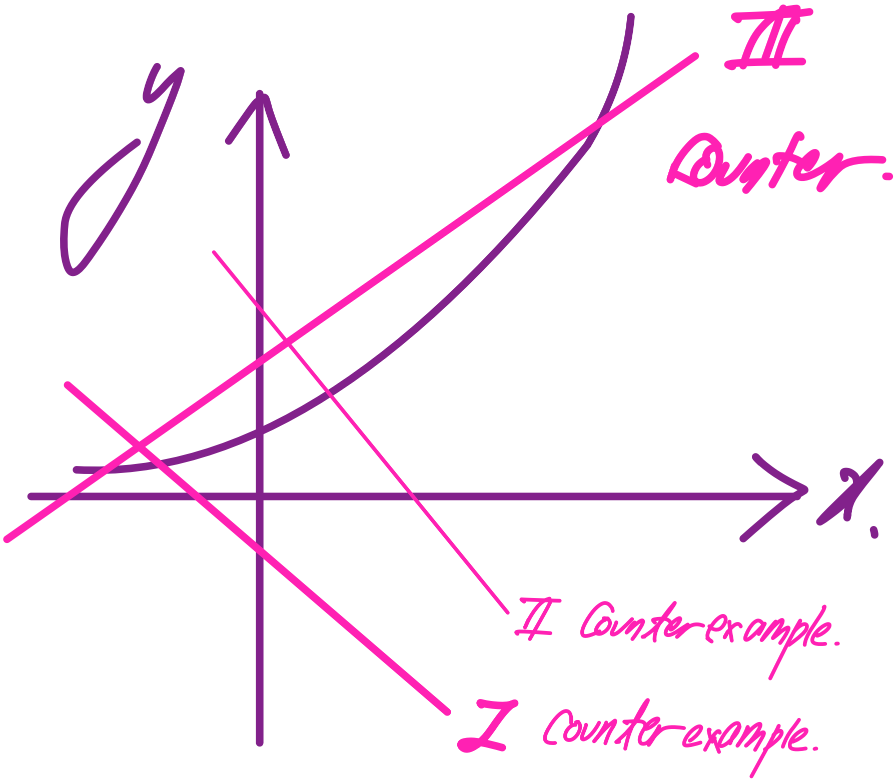
D III only

E I and II only

F I and III only

G II and III only

H I, II and III



* Statement Negation: Don't Overthink it, just Compartmentalise it & sub in all.

12 Consider the following statement:

> For any positive integer N there is a positive integer K such that $N(Km+1)-1$ is not prime for any positive integer m.

Which one of the following is the negation of this statement?

A For any positive integer N there is a positive integer K such that there is a positive integer m for which $N(Km+1)-1$ is prime.

B For any positive integer N there is a positive integer K such that there is a positive integer m for which $N(Km+1)-1$ is not prime.

C For any positive integer N there is a positive integer K such that for any positive integer m, $N(Km+1)-1$ is not prime.

D For any positive integer N, any positive integer K and any positive integer m, $N(Km+1)-1$ is not prime.

E There is a positive integer N such that for any positive integer K there is a positive integer m for which $N(Km+1)-1$ is not prime.

F There is a positive integer N such that for any positive integer K there is a positive integer m for which $N(Km+1)-1$ is prime.

G There is a positive integer N such that for any positive integer K and any positive integer m, $N(Km+1)-1$ is prime.

H There is a positive integer N and a positive integer K for which there is no positive integer m for which $N(Km+1)-1$ is prime.

Just because $x^2>1$ Doesn't mean $x>1$. x could be -2 billion. $\rightarrow$ $x<-1$ or $x>1$.

13 The following is an attempted proof of the conjecture:

$$\textbf{if } \tan\theta > 0, \textbf{ then } \sin\theta + \cos\theta > 1.$$

Suppose $\tan\theta > 0$, so in particular $\cos\theta \neq 0$.

Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then $\sin\theta\cos\theta = \tan\theta\cos^2\theta > 0$. (I)

It follows that $1 + 2\sin\theta\cos\theta > 1$. (II)

Therefore $\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta > 1$, (III)

which factorises to give $(\sin\theta + \cos\theta)^2 > 1$. (IV)

Therefore $\sin\theta + \cos\theta > 1$. (V)

$A^2 > B^2$.

$\rightarrow A^2 - B^2 > 0$

$(A-B)(A+B) > 0$

$(A-B, A+B)$

$= (+, +)$

$(-, -)$

Which one of the following is the case?

A The proof is correct.

B The proof is incorrect, and the first error occurs in line (I).

C The proof is incorrect, and the first error occurs in line (II).

D The proof is incorrect, and the first error occurs in line (III).

E The proof is incorrect, and the first error occurs in line (IV).

F The proof is incorrect, and the first error occurs in line (V).

"uniquely determines" = singular = only one

↗ 「Noir」 for Cosine rule expansion exists.

4. In the triangle PQR, $PR = 2$, $QR = p$ and $\angle RPQ = 30°$.

What is the set of **all** the values of p for which this information uniquely determines the length of PQ?

A $p = 1$

B $p = \sqrt{3}$

C $1 \leq p < 2$

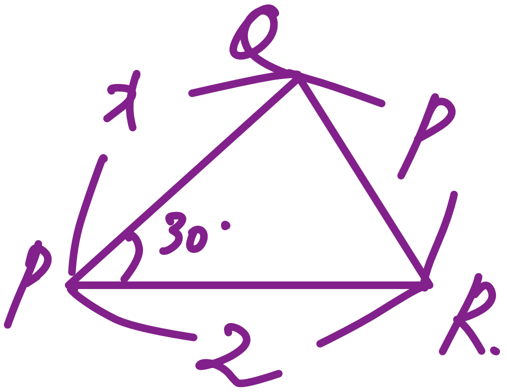
D $\sqrt{3} \leq p < 2$

E $p = 1$ or $p \geq 2$

F $p = \sqrt{3}$ or $p \geq 2$

G $p < 2$

H $p \geq 2$



9. Existence of triangles.

Confluences: 3 sides & 1 angle : Cosine rule.

① $\cos 30° = \dfrac{2^2 + x^2 - p^2}{2 \cdot 2 \cdot x}$

$\frac{\sqrt{3}}{2} \times 4x = x^2 - p^2 + 4$

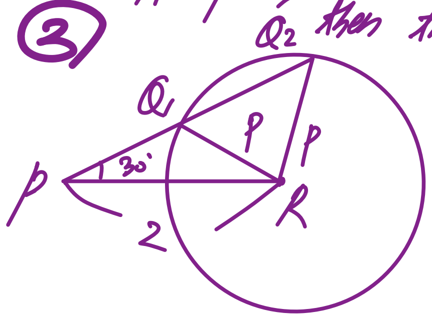
$x^2 - 2\sqrt{3}x - p^2 + 4 = 0.$

$D/4 \Rightarrow$ ↗ Use 「Noir」's half angle formulae.

$x = \sqrt{3} \pm \sqrt{3 + p^2 - 4}$

$p = 1$

② if $p<2$, then there are two possible triangles. ⇒ Hence, at $p \geq 2$, there are two triangles.



7. [Nbir] translation. $f(-2q) = -8q^3 + 12q^3 + 2 = 4q^3 + 2.$

15 It is given that $\mathrm{f}(x) = x^3 + 3qx^2 + 2$, where q is a real constant.

The equation $\mathrm{f}(x) = 0$ has 3 distinct real roots.

Which of the following statements is/are **necessarily** true?

I The equation $\mathrm{f}(x) + 1 = 0$ has 3 distinct real roots. Depends on q.

II The equation $\mathrm{f}(x + 1) = 0$ has 3 distinct real roots.

III The equation $\mathrm{f}(-x) - 1 = 0$ has 3 distinct real roots. likewise.

A none of them

B I only

C II only

D III only

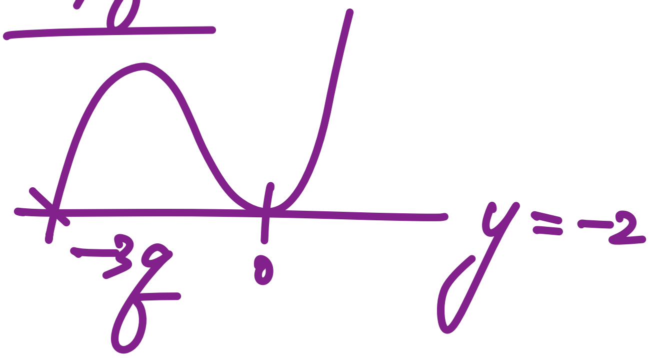
E I and II only

F I and III only

G II and III only

H I, II and III

i) $q > 0$

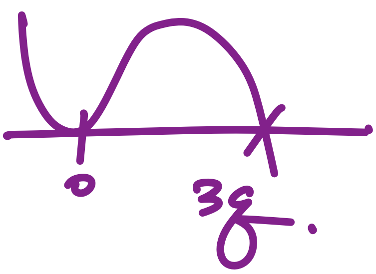


ii) $q = 0$ (X) Can't have 3 distinct REAL roots.

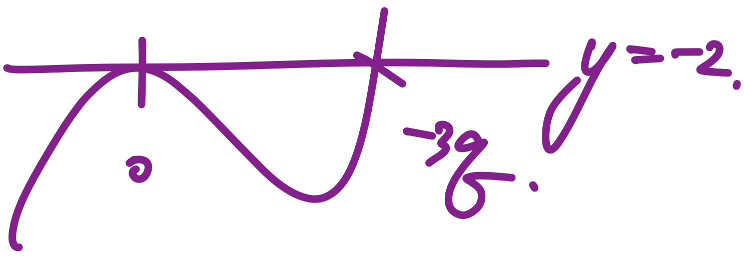
III: $f(-x) - 1$:

$-x^3 + 3qx^2 + 1.$

$-x^2(x - 3q) + 1.$



iii) $q < 0$



to minimise risk, you need an algebraic verification. Don't just vaguely trust your gut. lead with instinct but you gotta prove it.

16 In this question, x_1, x_2, x_3, $\ldots$ is an **arithmetic progression**, all of whose terms are integers.

Let n be a positive integer. If the median of the first n terms of the sequence is an integer, which of the following three statements **must** be true?

9. Algebraic Proof.

I The median of the first $n + 2$ terms is an integer.

~~II~~ The median of the first $2n$ terms is an integer.

III The median of x_2, x_4, x_6, $\ldots$, x_{2n} is an integer.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I. (median of first n terms) $\oplus$ next term = integer.

(that term might not exist itself in the sequence like $a_{4.5}$, but since $d \in \mathbb{Z}$, it is.)

~~II~~. $a_{\frac{n+1}{2}}$ integer $\not\rightarrow$ $a_{\frac{2n+1}{2}}$

III. x_n is median for III. $x_n \in \mathbb{Z}$.

17 A positive integer is called a *squaresum* **if and only if** it can be written as the sum of the squares of two integers. For example, 61 and 9 are both squaresums since $61 = 5^2 + 6^2$ and $9 = 3^2 + 0^2$.

A prime number is called *awkward* **if and only if** it has a remainder of 3 when divided by 4. For example, 23 is awkward since $23 = 5 \times 4 + 3$.

A (true) theorem due to Fermat states that:

> A positive integer is a squaresum **if and only if** each of its awkward prime factors occurs to an even power in its prime factorisation.

It follows that 5×23^2 is a squaresum, since 23 occurs to the power 2, but 5×23^3 is not, since 23 occurs to the power 3.

Which one of the following statements is **not** true?

A Every square number is a squaresum. O : $a^2 = a^2 + 0^2$ ∴ squaresum.

B If N and M are squaresums, then so is NM. O

C If NM is a squaresum, then N and M are squaresums. X split.

D If N is not a squaresum, then kN is a squaresum for some number k which is a product of awkward primes. O

B:

$N = (\ )^{2☆}$

$M = (\ )^{2\triangle}$

awkward primes

$\times$

$NM = (\ )^{\text{even}}$

D: $N = k_1^{\text{odd}} \times k_2^{\text{odd}} \times \cdots k_n^{\text{odd}} \times (\ )^{\text{even}}$

$k = k_1^{\text{odd}} \times k_2^{\text{odd}} \times \cdots k_n^{\text{odd}}$

Concavity # Trapezium Rule # AM. GM Rule

*It was a question of concavity.

18 f(x) is a polynomial function defined for all real x.

Which of the following is a **necessary** condition for the inequality

$$\frac{\mathrm{f}(a)+\mathrm{f}(b)}{2} \geq \mathrm{f}\left(\frac{a+b}{2}\right)$$

to be true for all real numbers a and b with $a < b$?

A $\mathrm{f}(x) \geq 0$ for all real x

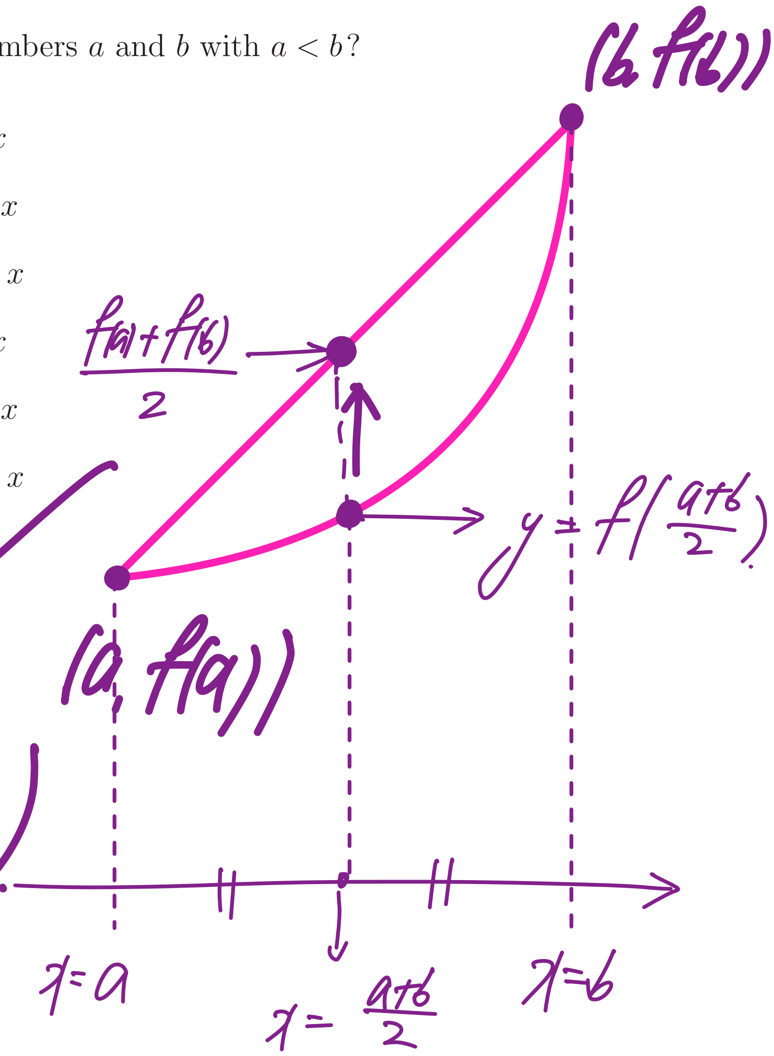
B $\mathrm{f}'(x) \geq 0$ for all real x

C $\mathrm{f}''(x) \geq 0$ for all real x

D $\mathrm{f}(x) \leq 0$ for all real x

E $\mathrm{f}'(x) \leq 0$ for all real x

F $\mathrm{f}''(x) \leq 0$ for all real x



$$\frac{f(a)+f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$$

19 Three **real** numbers x, y and z satisfy $x > y > z > 1$.

Which one of the following statements **must** be true?

D.

A $\dfrac{2^{z+1}}{2^x} > \dfrac{2^x + 2^z}{2^y}$

B $2 > \dfrac{3^x + 3^z}{3^y}$

C $\dfrac{2 \times 5^x}{5^z} > \dfrac{5^x + 5^z}{5^y}$

D $2 < \dfrac{7^x + 7^z}{7^y}$

(A) $\dfrac{2^z}{2^x} + \dfrac{2^x}{2^x} > \dfrac{2^x}{2^y} + \dfrac{2^z}{2^y}$

$\dfrac{2^z}{2^x} - \dfrac{2^z}{2^y} > \dfrac{2^x}{2^y} - \dfrac{2^x}{2^x}$

$2^z\left(\dfrac{1}{2^z} - \dfrac{1}{2^y}\right) > 2^x\left(\dfrac{1}{2^y} - \dfrac{1}{2^x}\right)$.

(B) $1+1 > \dfrac{3^x}{3^y} + \dfrac{3^z}{3^y}$.

$x > y > z > 1$ so

$\dfrac{3^x}{3^y} > 1$. & $\dfrac{3^z}{3^y} > 1$.

Hence, $\dfrac{3^x}{3^y} + \dfrac{3^z}{3^y} > 2$. (×)

(C) $\dfrac{5^x}{5^z} + \dfrac{5^x}{5^z} > \dfrac{5^x}{5^y} + \dfrac{5^z}{5^y}$

✓ $\dfrac{5^x}{5^z} > \dfrac{5^x}{5^y}$ $\left(\left(\frac{1}{5}\right)^z > \left(\frac{1}{5}\right)^y \right)$

As $z < y$ and $\left(\frac{1}{5}\right)^x$ is decreasing

✓ $\dfrac{5^x}{5^z} > \dfrac{5^z}{5^y}$.

↓ $5^x 5^y > 5^z 5^z$. (0)

($\because 5^x > 5^z > 0$, $5^y > 5^z > 0$)

(D) $\dfrac{7^x}{7^y} > 1$, $\dfrac{7^z}{7^y} < 1$

Hence. $\dfrac{7^x + 7^z}{7^y}$ can or cannot be > 2.

(not a "must").

★ "or" means union.

20 It is given that the equation $\sqrt{x+p} + \sqrt{x} = p$ has at least one real solution for x, where p is a real constant.

(B)

What is the complete set of possible values for p?

A $p = 0$ or $p = 1$

B $p = 0$ or $p \geq 1$

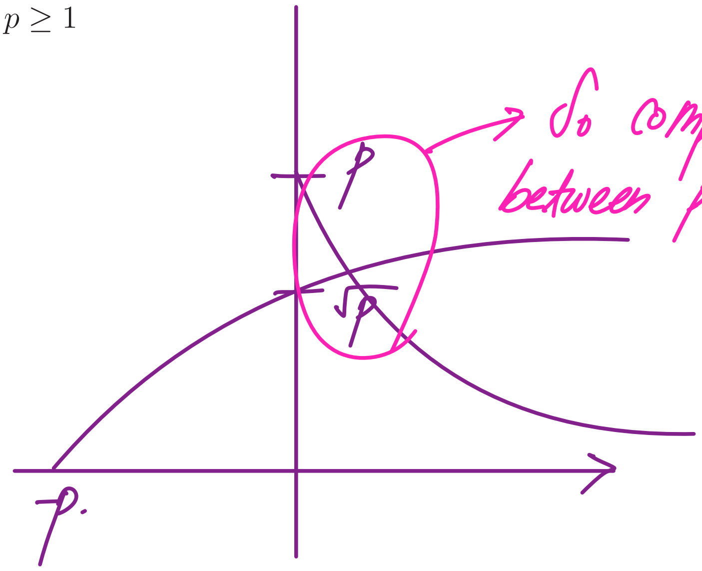
C $p \geq -x$

D $p \geq \sqrt{x}$

E $p \geq 0$

F $p \geq 1$

$\sqrt{x+p} = p - \sqrt{x}$.



So comparison of magnitude between p & √p.

$p \geq \sqrt{p}$.

$\sqrt{p} \times \sqrt{p} \geq \sqrt{p}$.

$\sqrt{p} \geq 1$.

$p \geq 1$. ~~$p \leq -1$~~

★ $\sqrt{p} \geq 0$.

& if $\sqrt{p} = 0$.

($p = 0$), then

there is real roots

so must be included

into total set of solutions.

END OF TEST

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TMUA 9.

Jamie Back

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