Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION **D513/01**

PAPER 1 **TMUA9. Model Answer.**

Wednesday 30 October 2019 **75 minutes**

Additional materials: Answer sheet

5152628418

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2

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TMUA 9.
Jamie Bark.

1 f(x) is a quadratic function in x.

The graph of $y = \mathrm{f}(x)$ passes through the point $(1, -1)$ and has a turning point at $(-1, 3)$.

Find an expression for f(x).

A $-x^2 - 2x + 2$

B $-x^2 + 2x + 3$

C $x^2 - 2x$

D $x^2 + 2x - 4$

E $2x^2 + 4x + 1$

F $-2x^2 - 4x + 5$

$f(x) = a(x+1)^2 + 3$

$f(1) = -1$

$f(1) = a \cdot 4 + 3 = -1$

$a = -1$

$\therefore f(x) = -(x+1)^2 + 3$

$= -x^2 - 2x + 2$

* for Quadratic to always have be positive ($a>0$)/ negative ($a<0$) s.t "a" is their leading coefficient, then $D<0$.

Not $D>0$ &

2 Find the complete set of values of the real constant k for which the expression

$$x^2 + kx + 2x + 1 - 2k$$

is positive for all real values of x.

A $-12 < k < 0$

B $k < -12$ or $k > 0$

C $-\sqrt{6} - 3 < k < \sqrt{6} - 3$

D $k < -\sqrt{6} - 3$ or $k > \sqrt{6} - 3$

E $-2 < k < \frac{1}{2}$

F $k < -2$ or $k > \frac{1}{2}$

G $0 < k < 4$

H $k < 0$ or $k > 4$

$x^2 + (k+2)x + 1 - 2k$

$D < 0$.

$D: (k+2)^2 - 4(1-2k)$

$= k^2 + 4k + 4 - 4 + 8k$

$= k^2 + 12k = k(k+12)$

$k(k+12) < 0$

$-12 < k < 0$

3 Find the coefficient of x in the expression:

E

$$(1+x)^0 + (1+x)^1 + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^{79} + (1+x)^{80}$$

A 80

B 81

C 324

D 628

E 3240

F 3321

G 6480

H 6642

$${}_{1}C_{1} + {}_{2}C_{1} + {}_{3}C_{1} + \cdots + {}_{80}C_{1}$$

$$= \sum_{k=1}^{80} k$$

$$= \frac{80 \times 81}{2}$$

$$= 40 \times 81 = 3240$$

4 The sequence x_n is given by:

$$x_1 = 10$$

$$x_{n+1} = \sqrt{x_n} \quad \text{for } n \geq 1$$

What is the value of x_{100}?

[Note that a^{b^c} means $a^{(b^c)}$]

A $10^{2^{99}}$

B $10^{2^{100}}$

C $10^{2^{-99}}$

D $10^{2^{-100}}$

E $10^{-2^{99}}$

F $10^{-2^{100}}$

G $10^{-2^{-99}}$

H $10^{-2^{-100}}$

keep the structure to make it easier for pattern recognition

at $n=1$: $x_2 = (x_1)^{\frac{1}{2}} = 10^{\left(\frac{1}{2}\right)^1}$

$n=2$: $x_3 = (x_2)^{\frac{1}{2}} = 10^{\left(\frac{1}{2}\right)^2}$

$\vdots$

$x_{100} = (x_{99})^{\frac{1}{2}} = 10^{\left(\frac{1}{2}\right)^{99}}$

$= 10^{2^{-99}}$

S_n = Quadratic function.

Use it &

5 S is a geometric sequence.

The sum of the first 6 terms of S is equal to 9 times the sum of the first 3 terms of S.

The 7th term of S is 360.

Find the 1st term of S.

$S_6 = 9S_3$ & $a_7 = 360$

A $\frac{40}{27}$

B $\frac{40}{9}$

C $\frac{40}{3}$

D $\frac{45}{16}$

E $\frac{45}{8}$

F $\frac{45}{4}$

Sol(1). Use $S_n = a \times \frac{r^n - 1}{r - 1}$ & difference of squares

$$\cancel{a} \cdot \frac{\overbrace{r^6 - 1}^{=(r^3-1)(r^3+1)}}{\cancel{r-1}} = 9\cancel{a} \cdot \frac{\cancel{r^3 - 1}}{\cancel{r-1}} \rightarrow r^3 + 1 = 9 \quad \therefore r^3 = 8$$

$$a = \frac{a_7}{r^6} = \frac{360}{64} = \frac{45}{8} \quad (\div 8)$$

£ Both Sol(1) & Sol(2) are great solutions.

Sol(2) Pure sequence attitude: Jot them out.

$$S_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \quad (\times r^3)$$

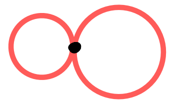
$$= (a_1 + a_2 + a_3) + r^3(a_1 + a_2 + a_3)$$

$$= (1 + r^3)(a_1 + a_2 + a_3)$$

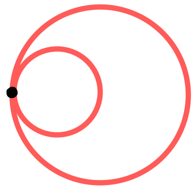
$$= (1 + r^3) S_3 = 9 S_3 \quad \therefore r^3 = 8.$$

$$a_7 = ar^6 = a \cdot (r^3)^2 = 64a = 360. \quad a = \frac{45}{8}$$

means either
① circumscribe



② inscribe



6 The circles with equations

$$(x+4)^2 + (y+1)^2 = 64 \quad \text{and}$$

$$(x-8)^2 + (y-4)^2 = r^2 \quad \text{where } r > 0$$

have exactly one point in common.

Find the difference between the two possible values of r.

A 4

B 10

C 16

D 26

E 50

$d = \sqrt{12^2 + 5^2} = 13$ (Distance between centres)

$r + 8 = 13$ (Circumscribe)

$r = 5$

$|r - 8| = 13$ (inscribe)

$r = 21$

$21 - 5 = 16.$

notice how the ideal solution not only does not sketch the x/y axis, not to mention the actual circle. Rare case of only needing algebra.

7 A curve has equation

$$y = (2q - x^2)(2qx + 3)$$

The gradient of the curve at $x = -1$ is a function of q.

Find the value of q which minimises the gradient of the curve at $x = -1$.

A -1

B $-\frac{3}{4}$

C $-\frac{1}{2}$

D 0

E $\frac{1}{2}$

F $\frac{3}{4}$

G 1

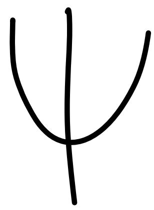
$$y' = -2x(2qx+3) + (2q - x^2) \times 2q.$$

$$y'\big|_{x=-1} = 2(-2q+3) + (2q-1)2q$$

$$= -4q + 6 + 4q^2 - 2q$$

$$= \underline{4q^2 - 6q + 6.}$$

$\min(4q^2 - 6q + 6)$ at q = vertex.



vertex: $\dfrac{-b}{2a} = \dfrac{6}{2 \times 4}$

$= \dfrac{3}{4}.$

cf). f(x) is concave up $\overset{\circ}{\underset{\times}{\rightleftharpoons}}$ overestimate

y=f(x) is concave up is sufficient but not necessary for an overestimate.

8 E

The function f is such that $0 < \mathrm{f}(x) < 1$ for $0 \leq x \leq 1$.

The trapezium rule with n equal intervals is used to estimate $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ and produces an underestimate.

Using the same number of equal intervals, for which one of the following does the trapezium rule produce an overestimate?

A $\int_0^1 (\mathrm{f}(x) + 1)\,\mathrm{d}x$

B $\int_0^1 2\mathrm{f}(x)\,\mathrm{d}x$

C $\int_{-1}^0 \mathrm{f}(x + 1)\,\mathrm{d}x$

D $\int_{-1}^0 \mathrm{f}(-x)\,\mathrm{d}x$

E $\int_0^1 (1 - \mathrm{f}(x))\,\mathrm{d}x$

* Don't even need to visualise this process.

sol idea) Instantly reading the question, I skimmed A to E to find a f(x)+b form s.t a<0

Why? because that means f(x) is reflected by a horizontal line.

↓

E has same interval but (negative constant) × f(x)

★ for Quadratics yes definitely.

9. Polynomials, Surds.

Inverse functions thus $y=x$ symmetric.

9 p is a positive constant.

Find the area enclosed between the curves $y = p\sqrt{x}$ and $x = p\sqrt{y}$

A $\frac{2}{3}p^{\frac{5}{2}} - \frac{1}{2}p^2$

B $\frac{4}{3}p^{\frac{5}{2}} - p^2$

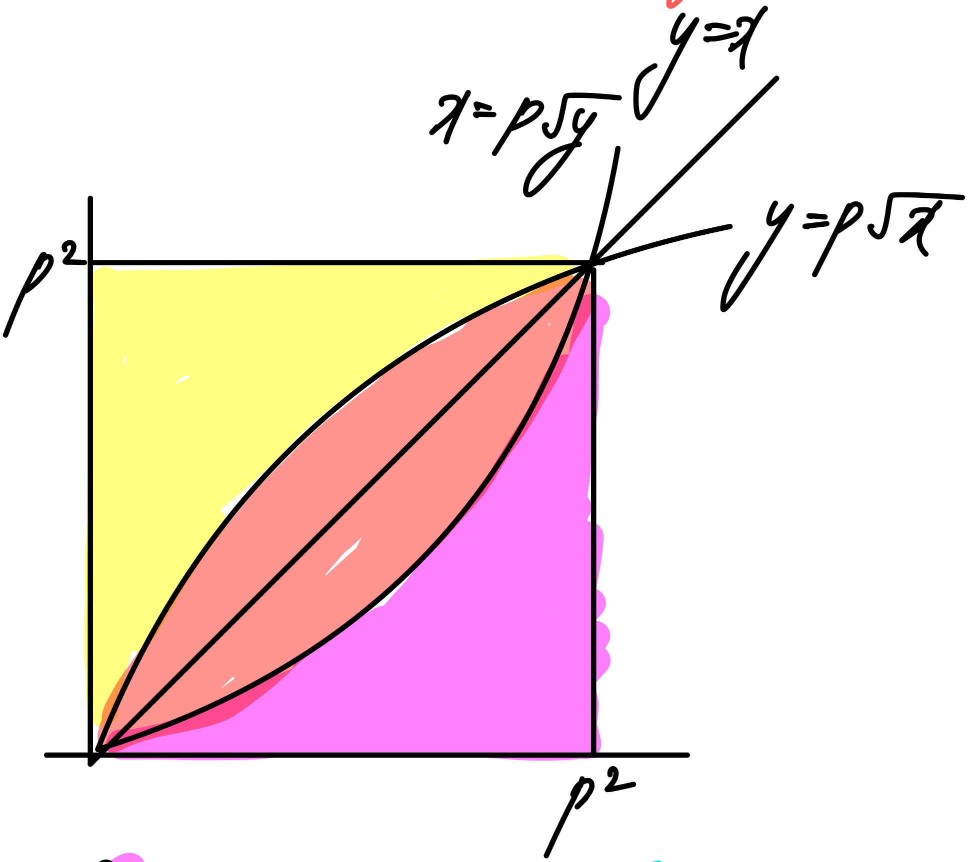
C $\frac{p^4}{6}$

D $\frac{p^4}{3}$

E $\frac{2}{3}p^3 - \frac{1}{2}p^4$

F $\frac{4}{3}p^3 - p^4$

G $2p^4$



$\int_1 : \int_2 : \int_3 = 1:1:1$ (Check Noir Premiere)

$\frac{p^4}{3}$.

☆ Piecewise functions: divide domains with inequalities

10 Evaluate

$$\int_{-1}^{3} |x|(1-x)\,\mathrm{d}x$$

A $\frac{17}{3}$

B $-\frac{17}{3}$

C $\frac{16}{3}$

D $-\frac{16}{3}$

E $\frac{11}{3}$

F $-\frac{11}{3}$

let $f(x) = |x|(1-x)$

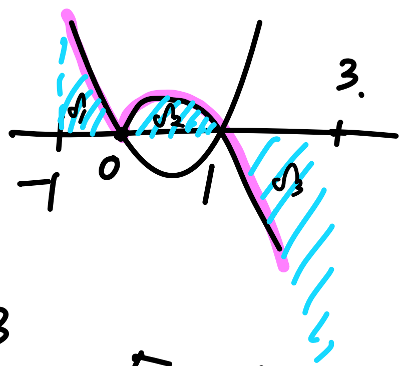
i) $x \geq 0$

$f(x) = x(1-x) = -x(x-1)$

ii) $x < 0$

$f(x) = -x(1-x) = x(x-1)$

$\downarrow$



$S_1 + S_2 + S_3$

$S_1 = \int_{-1}^{0} (x^2 - x)\,dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{0}$

$= 0 - \left(-\frac{1}{3} - \frac{1}{2}\right)$

$= \frac{5}{6}$

$\int_{-1}^{3} = \frac{5}{6} + \frac{1}{6} - \frac{14}{3} = -\frac{11}{3}$

$S_2 = \frac{1}{6} \times 1^3 = \frac{1}{6}$ (See Noir Premeire)

$S_3 = \int_{1}^{3} (x^2 - x)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{1}^{3} = \frac{1}{3}(27-1) - \frac{1}{2}(9-1) = \frac{14}{3}$

11 Find the sum of the real values of x that satisfy the simultaneous equations:

habituate explicitly writing domains when introducing new domains

$$\log_3(xy^2) = 1$$

$$(\log_3 x)(\log_3 y) = -3$$

A $\frac{1}{3}$

B 1

C 3

D $3\frac{1}{9}$

E $9\frac{1}{27}$

F $9\frac{1}{3}$

G 27

H $27\frac{1}{9}$

let $\log_3 x = X$, $\log_3 y = Y$ $(X, Y \in \mathbb{R})$

$X + 2Y = 1$

$XY = -3$

$X = 1 - 2Y$

$XY = Y(1-2Y) = -3$

$2Y^2 - Y - 3 = 0$

$\begin{matrix} 2 & -3 \\ 1 & 1 \end{matrix}$

$(2Y-3)(Y+1) = 0$

$Y = \frac{3}{2}, -1$

$X = -2, 3$

$\log_3 x = -2, 3$

$x = 3^{-2}, 3^3$

$27\frac{1}{9}$

$t-1=(\sqrt{t}-1)(\sqrt{t}+1)$
(Differences of squares)

9. Algebra.

12 It is given that

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{24\pi(t-1)}{(1+\sqrt{t}\,)}\quad \text{for } t\geq 1$$

and $V=7$ when $t=1$.

Find the value of V when $t=9$.

A $208\pi+7$

B $216\pi+7$

C $224\pi+7$

D $416\pi+7$

E $608\pi+7$

F $744\pi+7$

RHS: $\dfrac{24\pi(\sqrt{t}-1)\cancel{(\sqrt{t}+1)}}{\cancel{(\sqrt{t}+1)}}$

$$\frac{dV}{dt}=24\pi(\sqrt{t}-1).$$

$\int$

$$V=\int 24\pi(t^{\frac{1}{2}}-1)\,dt$$

$$=24\pi\int(t^{\frac{1}{2}}-1)\,dt$$

$$\therefore V=24\pi\left(\frac{2}{3}t^{\frac{3}{2}}-t\right)+C$$

at $t=1$, $V=7$.

$$24\pi\left(\frac{2}{3}-1\right)+C=-8\pi+C=7$$

$$C=8\pi+7.$$

$$V(t=9)=24\pi\left(\frac{2}{3}\times27-9\right)+8\pi+7=(24\times9+8)\pi+7.$$

13 Find the maximum value of

$$4^{\sin x} - 4 \times 2^{\sin x} + \frac{17}{4}$$

for real x.

A $\frac{1}{4}$

B $\frac{5}{2}$

C $\frac{13}{2}$

D $\frac{21}{2}$

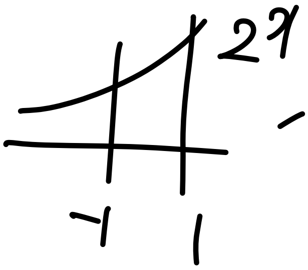
E $\frac{65}{4}$

F There is no maximum value.

let $2^{\sin x} = t$.

as $\sin x \in [-1, 1]$,

Specify Domain!

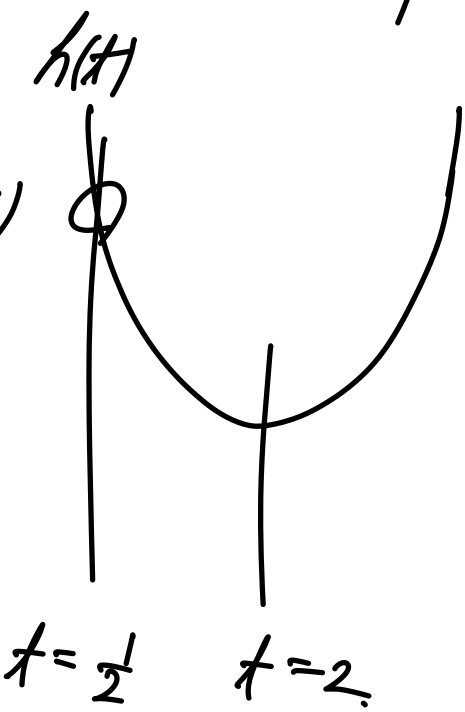


$(2^{-1} \leq t \leq 2^{1})$

$t^2 - 4t + 4 + \frac{1}{4}$

let $h(t) = (t-2)^2 + \frac{1}{4}$

max(h(t)) at $h(\frac{1}{2})$



$h(\frac{1}{2}) = (\frac{1}{2} - 2)^2 + \frac{1}{4}$

$= \frac{10}{4} = \frac{5}{2}$

important Q

14 x satisfies the simultaneous equations

$$\sin 2x + \sqrt{3}\cos 2x = -1$$

and

$$\sqrt{3}\sin 2x - \cos 2x = \sqrt{3}$$

where $0° \leq x \leq 360°$.

Specify Domain!

Find the sum of the possible values of x.

A 210°

B 330°

C 390°

D 660°

E 780°

F 930°

let $\sin 2x = S$
$\cos 2x = C$ $\quad (S, C \in [-1, 1])$.

$$\begin{cases} S + \sqrt{3}C = -1 \\ \sqrt{3}S - C = \sqrt{3}. \end{cases}$$

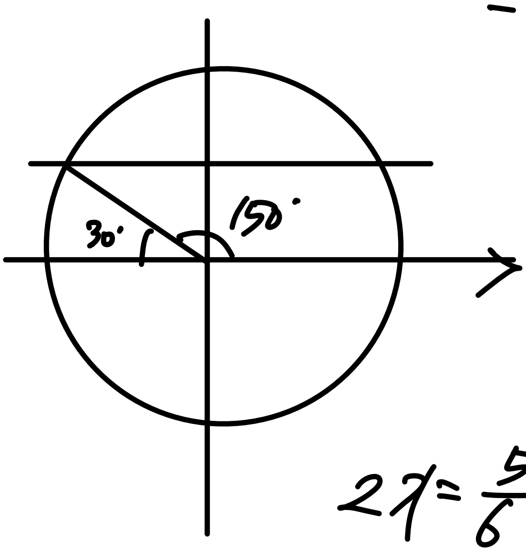
$S = -1 - \sqrt{3}C.$

$\sqrt{3}S - C = \sqrt{3}(-1 - \sqrt{3}C) - C$

$= -4C - \sqrt{3} = \sqrt{3}$

$\therefore C = \frac{-\sqrt{3}}{2}$

$S = -1 + \frac{3}{2} = \frac{1}{2}$



$2x = \frac{5}{6}\pi + 2n\pi$

$x = \frac{5}{12}\pi + n\pi \; (n \in W), \quad 0 \leq x \leq 2\pi.$

$x = \frac{5}{12}\pi, \frac{17}{12}\pi, \quad \Sigma x = \frac{22}{12}\pi = \frac{11}{6}\pi$

9. tools of exponents.

15 Find the real non-zero solution to the equation

$$\frac{2^{(9^x)}}{8^{(3^x)}} = \frac{1}{4}$$

$$\frac{2^{(3^x)^2}}{(2^3)^{3^x}} = \frac{2^{(3^x)^2}}{2^{(3^{x+1})}} = 2^{-2}$$

let $3^x = k$ $(k>0)$

$$2^{k^2-3k} = 2^{-2}.$$

A $\log_3 2$

B $2\log_3 2$

C 1

D 2

E $\log_2 3$

F $2\log_2 3$

$a=b \xrightarrow{\circ} f(a)=f(b)$

$a=b \xleftarrow{\times} f(a)=f(b)$

for "if $f(a)=f(b)$, then $a=b$" to be true, $f(x)$ must be one-to-one function.

As 2^x is one-to-one function,

$k^2-3k=-2.$

$\therefore k=1, 2.$

$3^x = 1, 2.$

$x=0, \underline{\log_3 2}$

9. Split the integration interval.

16 Given that

$$2\int_0^1 \mathrm{f}(x)\,\mathrm{d}x + 5\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = 14$$

30

and

$$\int_0^1 \mathrm{f}(x+1)\,\mathrm{d}x = 6$$

find the value of

$$\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$$

A -8

B -4

C -2

D 2

E 4

F $\frac{29}{5}$

G $\frac{32}{5}$

H 14

$\int_1^2 f = 6$

$2\int_0^1 + 30 = 14$

$\therefore \int_0^1 = -8.$

$\int_0^1 + \int_1^2 = -8+6$

CASE Division. Overcome the fear of complexity when dividing cases.

9. finding intersections for inequalities.

17 Find the fraction of the interval $0 \leq \theta \leq \pi$ for which the inequality

$$\underbrace{\left(\sin(2\theta) - \tfrac{1}{2}\right)}_{☆} \underbrace{\left(\sin\theta - \cos\theta\right)}_{\triangle} \geq 0$$

is satisfied.

A $\frac{1}{12}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{5}{12}$

E $\frac{7}{12}$

F $\frac{3}{4}$

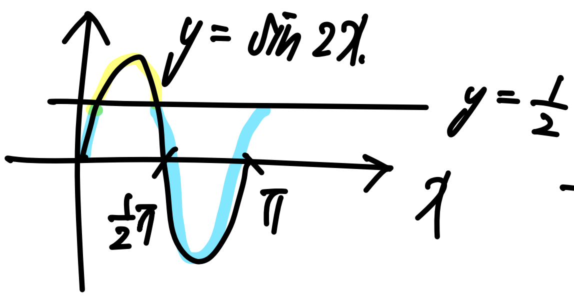
G $\frac{5}{6}$

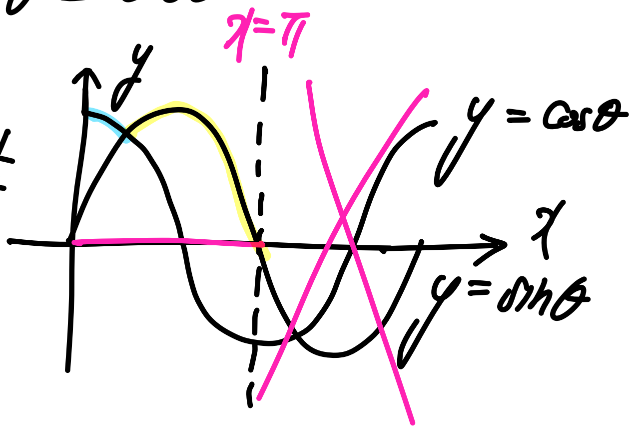
H $\frac{11}{12}$

(i) $☆ \geq 0$, $\triangle \geq 0$

$\sin 2\theta - \frac{1}{2} \geq 0$, $\sin\theta - \cos\theta \geq 0$

$\sin 2\theta \geq \frac{1}{2}$. $\sin\theta \geq \cos\theta$.





(ii) $☆ \leq 0$, $\triangle \leq 0$

$\sin 2\theta \leq \frac{1}{2}$. $\sin\theta \leq \cos\theta$.

(i) $\left(\frac{1}{12}\pi \leq \theta \leq \frac{5}{12}\pi\right) \cap \left(\frac{1}{4}\pi \leq \theta \leq \pi\right) = \frac{1}{4}\pi \leq \theta \leq \frac{5}{12}\pi$

→ interval length $= \frac{5}{12}\pi - \frac{1}{4}\pi = \frac{2}{12}\pi$.

(ii) $\left(0 \leq \theta \leq \frac{1}{12}\pi \text{ or } \frac{5}{12}\pi \leq \theta \leq \pi\right) \cap \left(0 \leq \theta \leq \frac{1}{4}\pi\right) = 0 \leq \theta \leq \frac{1}{12}\pi$.

interval length $= \frac{1}{12}\pi$

total =

sum $= \frac{3}{12}\pi = \frac{1}{4}\pi$

9. See next page for what was done here!

18 Find the shortest distance between the curve $y = x^2 + 4$ and the line $y = 2x - 2$.

A 2

B $\sqrt{5}$

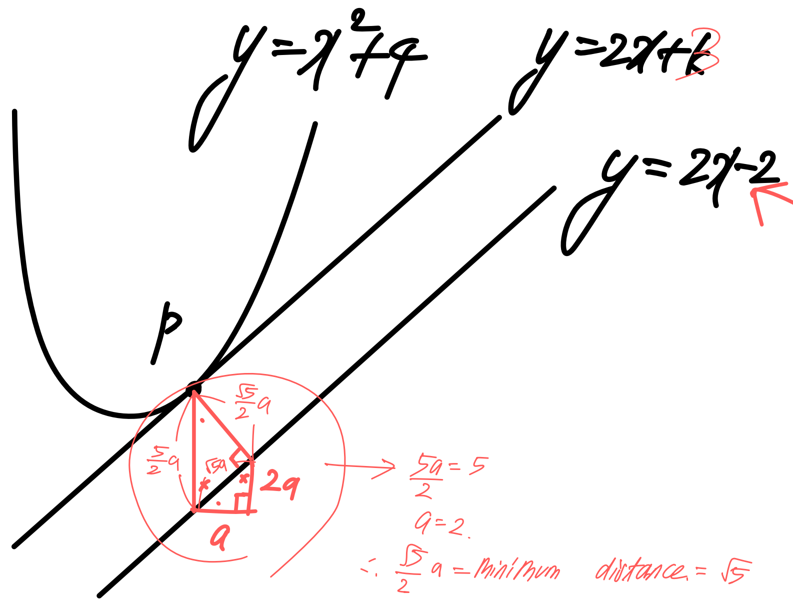
C $\frac{6\sqrt{5}}{5}$

D 3

E $\frac{5\sqrt{5}}{3}$

F 5

G 6



$\frac{5a}{2} = 5$

$a = 2$

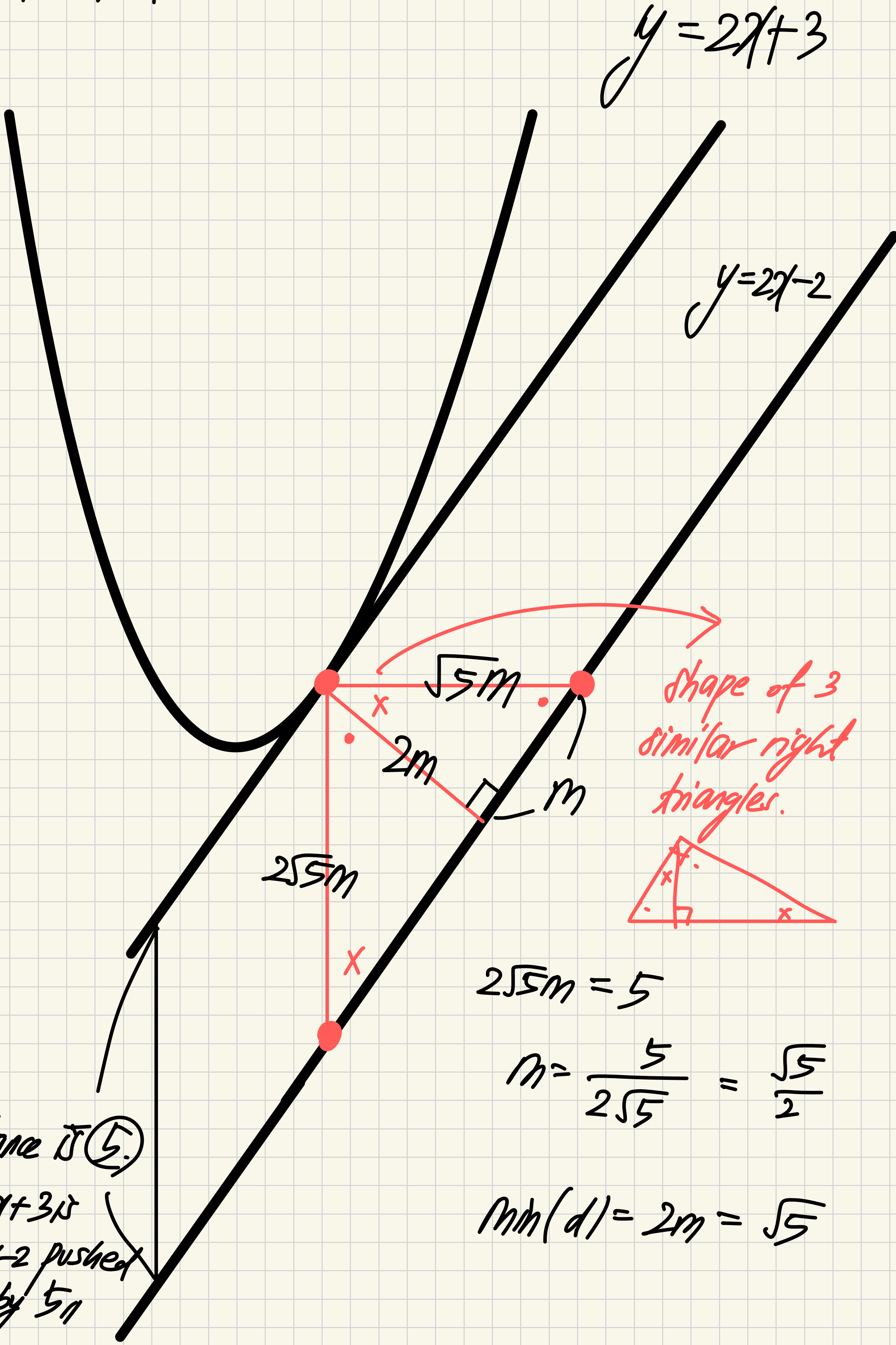
$\therefore \frac{\sqrt{5}}{2}a =$ minimum distance $= \sqrt{5}$

$y' = 2x = 2$ at $x=1$

$\therefore P(1, 1^2+4) = P(1,5)$

as it's on $y=2x+k$, $2(1)+k=5$, $\underline{k=3}$

9. Sol 2) (Ideal).



Distance is 5.

$y = 2x+3$ is $y=2x-2$ pushed up by 5

$$2\sqrt{5}m = 5$$

$$m = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$$

$$\min(d) = 2m = \sqrt{5}$$

9. Cyclic summations for trig on unit circle & applications.

19 Find the value of

$$\sum_{k=0}^{90} \sin(10+90k)^\circ$$

A 0

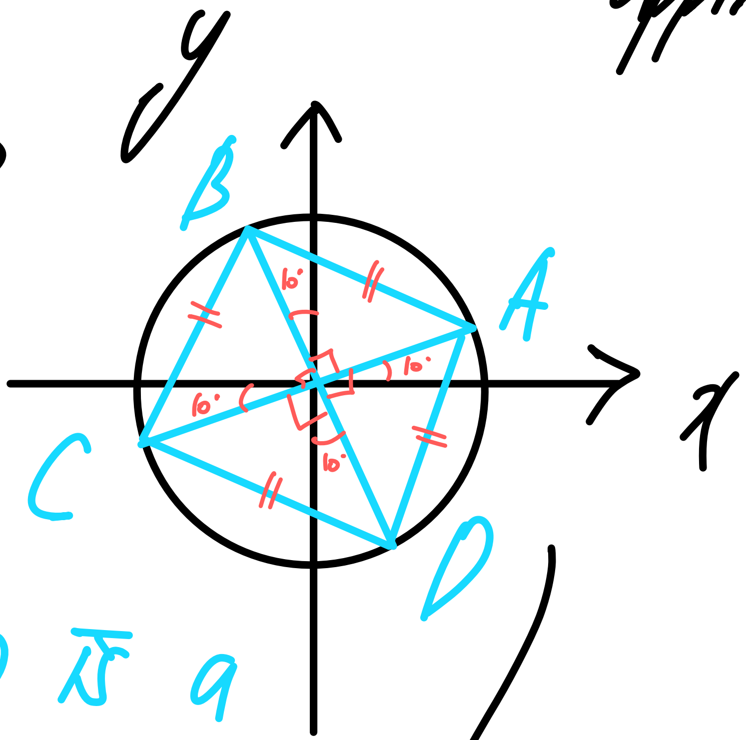
B $\sin 10^\circ$

C $\sin 100^\circ$

D $\sin 190^\circ$

E $\sin 280^\circ$

F 1



□ABCD is a square.

$\sin A + \sin C = 0.$

$\sin B + \sin D = 0.$

$\therefore 91 = 92 - 1$

$= 88 + 3.$

$$\therefore \sum_{k=0}^{90} \sin(10+90k)^\circ = \sin 10^\circ + \sin 100^\circ + \sin 190^\circ$$

$$= \sin 100^\circ.$$

20 What is the complete range of values of k for which the curves with equations

$$y = x^3 - 12x$$

and

$$y = k - (x-2)^2$$

intersect at **three** distinct points, of which exactly **two** have positive x-coordinates?

A $-4 < k < 0$

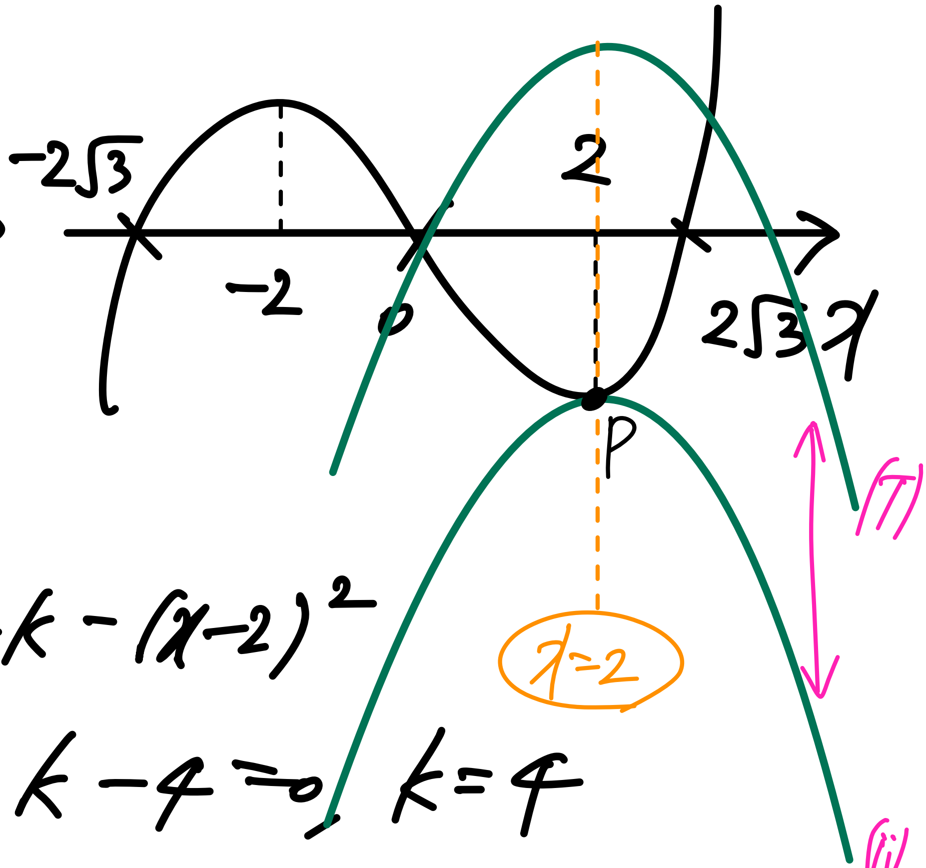
B $-4 < k < 4$

C $-4 < k < 16$

D $-16 < k < 0$

E $-16 < k < 4$

F $-16 < k < 16$



(i): $(0,0)$ $y = k - (x-2)^2$
$= k - 4 = 0$, $k = 4$

(ii) Local minimum of cubic = vertex of quadratic

$2^3 - 12 \times 2 = 8 - 24 = -16$

$\therefore k - (2-2)^2 = k = -16$

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Jamie Baek.

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