TMUA9. Model Answer.

Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION **D513/02**

PAPER 2

Wednesday 30 October 2019 **75 minutes**

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the second of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2


P.S. my cousin Sienna drew this so shoutout Sienna Jung (she's 10 btw).
Jamie
King!
I will draw my cousin Jamie <3
NBA.
TMUA9.
23
Art by: Sienna.

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TMUA 9.
Jamie Baek (백승주)

1 Find the coefficient of the x^4 term in the expansion of

$$x^2\left(2x+\frac{1}{x}\right)^6$$

$\boxed{\text{Sol X.}}$

A 15

B 30

C 60

D 120

E 240

$x^2 \times$ ___ x^2.

$\downarrow$

$(2x)^A \cdot \left(\frac{1}{x}\right)^B \cdot {}_6C_A$

s.t $A-B=2$

$A+B=6$.

$\rightarrow (A,B)=(4,2)$

$\therefore x^2 \times (2x)^4 \left(\frac{1}{x}\right)^2 \cdot {}_6C_4 \cdot {}_2C_2$.

$= x^2 \times 16x^2 \times 15$

$= x^2 \times 4x^2 \times 60.$

$= \underline{240x^4}$

$\boxed{\text{Sol 9.}}$ Pure intuition!

$x^2 \times \sim x^2 = x^4.$

so we're finding the term s.t x^2 within $\left(2x+\frac{1}{x}\right)^6$,

Obvious $\left(\begin{array}{l} 2+4=6 \\ \underline{4-2=2} \end{array} \right.$ it's okay to not be able to articulate it in words. Intuition isn't verbal.

★ Simple factor theorem.

2 $(2x+1)$ and $(x-2)$ are factors of $2x^3 + px^2 + q$.

C

What is the value of $2p + q$?

A -10

B $-\frac{38}{5}$

C $-\frac{22}{3}$

D $\frac{22}{3}$

E $\frac{38}{5}$

F 10

$f(x) = 2x^3 + px^2 + q.$

$f(-\frac{1}{2}) = f(2) = 0.$

$$\begin{cases} 2(-\frac{1}{2})^3 + p(-\frac{1}{2})^2 + q = 0 & \cdots ① \\ 2\cdot 2^3 + p\cdot 2^2 + q = 0. & \cdots ② \end{cases}$$

$\hookrightarrow q = -4p - 16.$

$2p + q = -2p - 16.$

*

Don't multiply integers just for the sake of fixing fractions into integer coefficients.

② − ①

$16 - 2(-\frac{1}{2})^3 + p(4 - \frac{1}{4}) = 0.$

$16 - 2\times(-\frac{1}{8}) + p(\frac{15}{4}) = 0.$

$p(-\frac{15}{4}) = \frac{65}{4}$

$p = -\frac{4}{15} \times \frac{65}{4} = -\frac{13}{3}$

$-2p - 16 = \frac{26}{3} - 16 = \frac{-22}{3}.$

「Reframe」
If $ab=ac$, then p is true.
from I~III, which can be p?

necessary condition

3 a, b and c are real numbers.

Given that $ab = ac$, which of the following statements **must** be true?

~~I~~ $a = 0$

~~II~~ $b = 0$ **or** $c = 0$

~~III~~ $b = c$

$ab = ac.$

$a(b-c) = 0.$

if $AB=0$, $\{A=0\} \cup \{B=0\}$.

(in verbal expressions: A or B is 0).

i) $a=0$ or ii) $b-c=0$

thus $b=c$.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

☆ Algorithm for solving these counter example.

⓪ split into condition & statement
① See if condition is true in the "candidates" for the counter examples
② key question: Is the 2nd half (=statement) ... (reinforcing) or (countering) the claim?

4 Consider the following conjecture:

> **If** N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1, **then** N is a prime number.

Here are three numbers:

I $N = 101$ (which is a prime number)

~~II~~ $N = 1001$ (which equals $7 \times 11 \times 13$) → Condition isn't true.

III $N = 10001$ (which equals 73×137)

Which of these provide(s) a counterexample to the conjecture?

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

[Ideal thought process]

We're finding counter example.

↓

① Condition is true.
② Statement is false.

↓

I & III meets ①.

② I reinforces given claim.
III disproves it.

* Do not be vague / common sense your way out of these type of questions.

Logic Questions necessitate a clear algorithm for Analysis. ⊕ Objective Judgement within that algorithm.

☆ Common attitude for solving logic questions; (See the structure/meaning)

- If given in CONCRETE (= specific examples) ⟷ Swing to the ABSTRACT.
- If given in ABSTRACT (= general terms either verbal/mathematical) ⟷ Swing to the CONCRETE. (= try examples!)

5 Consider the following statement about the positive integers a, b and n:

$$(*): \ ab \text{ is divisible by } n$$ → Given in ABSTRACT

Swing to Concrete.

$4 \mid 2 \times 8$ but $4 \nmid 2$.

The condition $\underbrace{\text{‘either } a \text{ or } b \text{ is divisible by } n\text{’}}_{}$ is: = ☆

A **necessary** but **not sufficient** for $(*)$

B **sufficient** but **not necessary** for $(*)$

C **necessary** and **sufficient** for $(*)$

D **not necessary** and **not sufficient** for $(*)$

Algorithm: Judge both. ($p \to q$ & $q \to p$.) & decide.

let given condition = ☆

If ☆ → then $*$ (o). ($\because$ either a or $b = nk$ s.t. $n, k \in \mathbb{Z}$)

$ab = nk \times (\text{integer})$

$= n \times (\text{INTEGER})$

If $*$ → then ☆. (X). find counter!

9. Attitude for logic questions.

→ Be Cocky (no Diddy) / Skeptical.

The default must be assuming they are all wrong.
IF, you end up having to admit it's true, there MUST be an objective vindication.

" Your inability to find counterexample is not an excuse to just say it's right."

similar question where $a^3 = (c-b)(c^2+cb+b^2)$ so $a = c-b$ and $a^2 = c^2+cb+b^2$

$\rightarrow$ prime factorisation $\neq$ algebraic factorisation.

6 A student attempts to solve the equation

$$\cos x + \sin x \tan x = 2 \sin x - 1$$

in the range $0 \leq x \leq 2\pi$.

The student's attempt is as follows:

$\cos x + \sin x \tan x = 2 \sin x - 1$

So $\cos x - \sin x + \sin x \tan x - \sin x = -1$ (I)

So $(\sin x - \cos x)(\tan x - 1) = -1$ (II)

So $\sin x - \cos x = -1$ or $\tan x - 1 = -1$ (III)

So $(\sin x - \cos x)^2 = 1$ or $\tan x = 0$ (IV)

So $2 \sin x \cos x = 0$ or $\tan x = 0$ (V)

So $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (VI)

Which of the following best describes this attempt?

A It is completely correct

B It is incorrect, and the first error occurs on line (I)

C It is incorrect, and the first error occurs on line (II)

D It is incorrect, and the first error occurs on line (III)

E It is incorrect, and the first error is that extra solutions were introduced on line (IV)

F It is incorrect, and the first error is that extra solutions were introduced on line (V)

G It is incorrect, and the first error is not eliminating the values where $\tan x$ is undefined on line (VI)

9.

Same algorithm as #4. ① Split it into if-/then-
② If- : is condition true?
③ then- : is statement reinforce / countering.

7 For which one of the following statements can the fact that $12^2 + 16^2 = 20^2$ be used to produce a **counterexample**?

~~A~~ If a, b and c are positive integers which satisfy the equation $a^2 + b^2 = c^2$, and the three numbers have no common divisor, then two of them are odd and the other is even. → gcd (12, 16, 20) = 4. So ① isn't met.

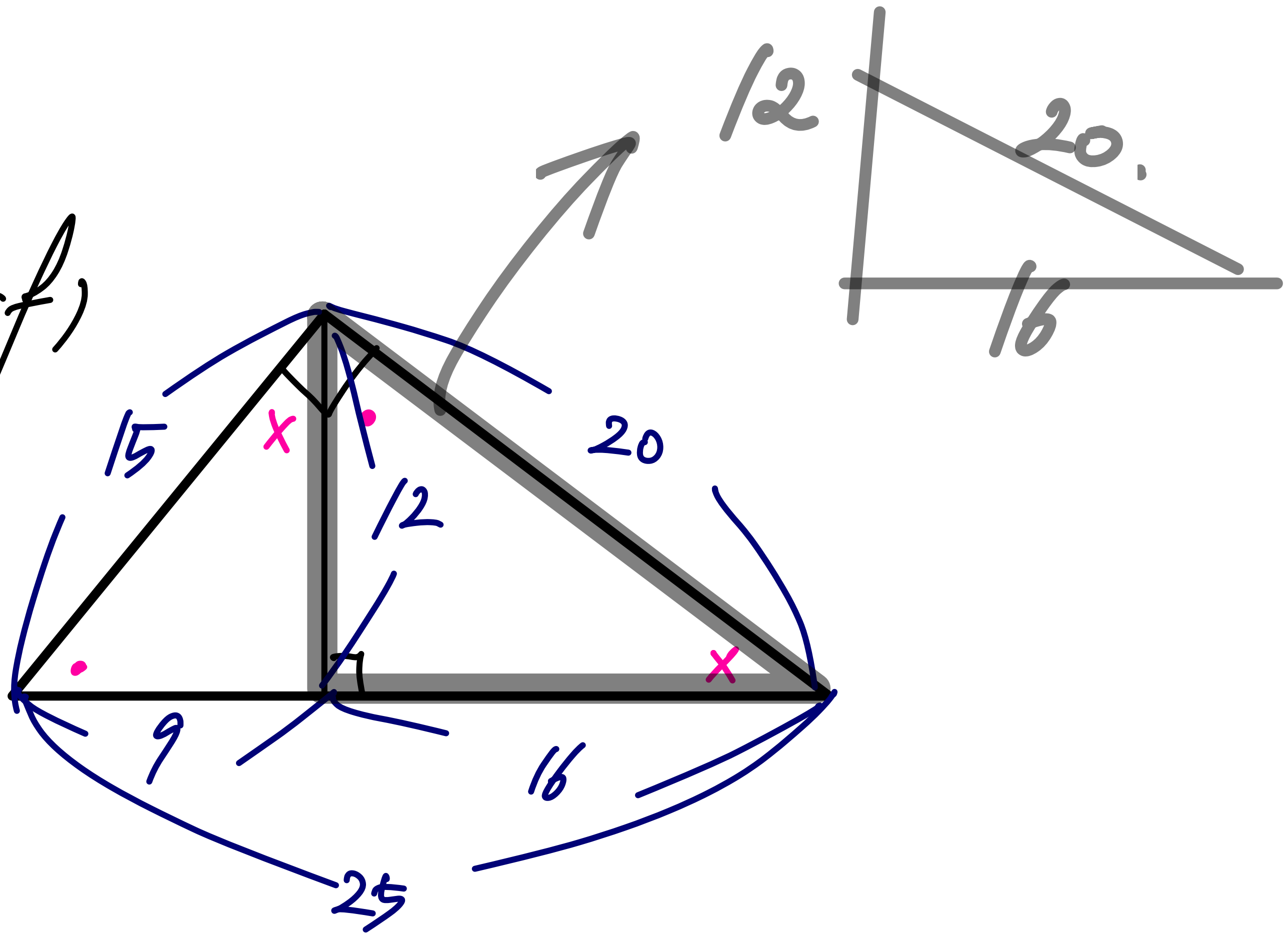
B The equation $a^4 + b^2 = c^2$ has no solutions for which a, b and c are positive integers. $4^4 + 12^2 = 20^2$ → ① ok) it is calm (apparently this is what Brits say). solution!

~~C~~ The equation $a^4 + b^4 = c^4$ has no solutions for which a, b and c are positive integers. $12^2 + 16^2 = 20^2$ isn't this form.

~~D~~ If a, b and c are positive integers which satisfy the equation $a^2 + b^2 = c^2$, then one is the arithmetic mean of the other two.

① ok → ② reinforces. So D supports claim.

(f)



8 a, b and c are real numbers with $a < b < c < 0$

Which of the following statements **must** be true?

I $ac < ab < a^2$

II $b(c + a) > 0$

III $\dfrac{c}{b} > \dfrac{a}{b}$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I : $a < b < c$

$\downarrow \times a$ ($= \ominus$ number)

$a^2 > ab > ac$

II : $a < b < c < 0$.

$ab > 0$. $bc > 0$.

$\therefore ab + bc > 0$

III. $\frac{1}{b} < 0$.

$a < c$

$\downarrow \times \left(\frac{1}{b}\right.$ which is negative number)

$\frac{a}{b} > \frac{c}{b}$.

* Caveats

(Multiplying the reciprocal of a negative number.)

$\neq$ (Taking the reciprocal of two numbers.)

★[Pigeonhole (No Diddy) Principle]: Read into it! It shows up in Oxbridge interviews.

★ Think of the flip side; when they don't.

9 A large circular table has 40 chairs round it.

What is the smallest number of people who can be sitting at the table already such that the next person to sit down **must** sit next to someone?

$40 = 3 \times 13 + 1$

A 9

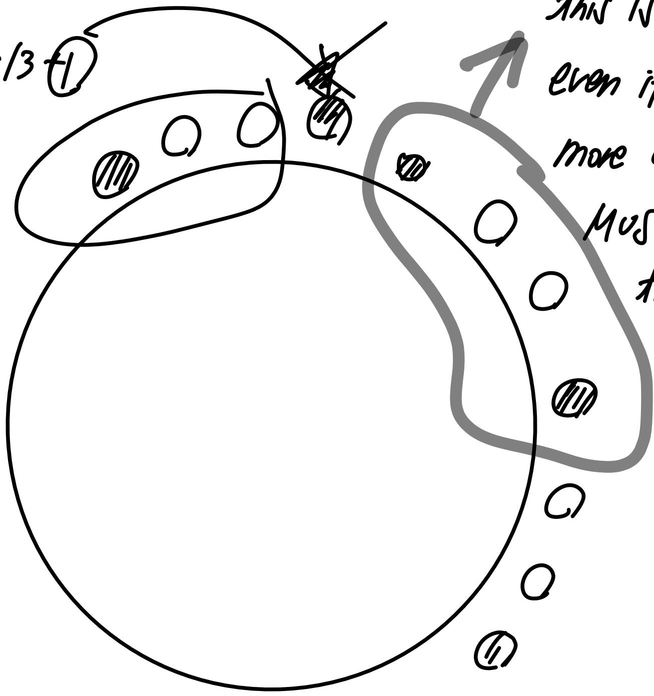
B 10

C 13

D 14

E 19

F 20



this is one cycle. because even if we add one more empty slot, someone MUST sit next to the two!

Prone to thinking this is the only case as it's most intuitive.

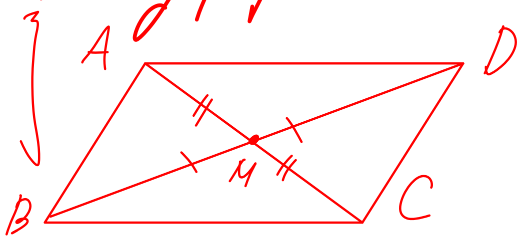
→ simple alteration...

Geometric Definition of Parallelogram & 5 key properties.

① $\overline{AB} = \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$ ④ $\overline{MA} = \overline{MC}$ and $\overline{MB} = \overline{MD}$

② $\overline{AB} = \overline{CD}$ and $\overline{AD} = \overline{BC}$ ⑤ $\angle ABC = \angle ADC$, $\angle BAD = \angle BCD$

③ $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$



10 $PQRS$ is a quadrilateral, labelled anticlockwise.

Which one of the following is a **necessary** but **not sufficient** condition for $PQRS$ to be a parallelogram?

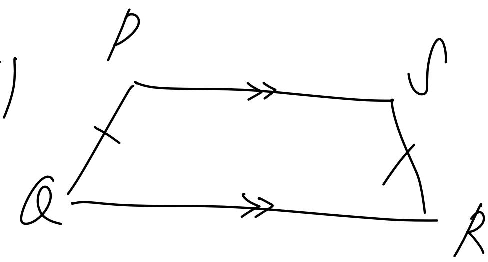
A $PQ = SR$ and PS is parallel to QR

~~**B**~~ $PQ = SR$ and PQ is parallel to SR

~~**C**~~ $PQ = QR = SR = PS$

~~**D**~~ $PR = QS$

(A)



Counter: Equidistant trapezoid. (Do Brits call this a trapezium?)

(B) Necessary & Sufficient! Identical

(C) Not even necessary at all. (C is definition of Rhombus)

(D) Likewise. not even necessary for parallelogram.

* You can solve it either with abstract(=generalised) form, or just concrete (finding counterexamples).

11 An **arithmetic** series has n terms, all of which are **integers**.

The sum of the series is 20.

Which of the following statements **must** be true?

I ~~The first term of the series is even.~~ Counter: 9, 11

II ~~n is even.~~ Counter: 20 (one term)

III The common difference is even.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Counter of III.

$$\frac{2a+(n-1)d}{2} \times n = 20.$$

looking at dn, d being even would help, but n could just be a multiple of 8.

$$4(2a+(n-1)d) = 20.$$

$$a_1 + a_n = 5.$$

So $a_1 + a_8 = 5$ & total of 8 terms.

$$2a + 7d = 5$$

try $a=-1$. $d=1$.

$$\underbrace{-1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}$$

12 Most students in a large college study Mathematics. A teacher chooses three different students at random, one after the other.

Order of selection is taken into consideration.

Consider these three probabilities:

$R = \text{P(At least one of the students chosen studies Mathematics)}$
$S = \text{P(The second student chosen studies Mathematics)}$
$T = \text{P(All three of the students chosen study Mathematics)}$

Which of the following is true?

A $R < S < T$

B $R < T < S$

C $S < R < T$

D $S < T < R$

E $T < R < S$

F $T < S < R$

R: 1 − P(all 3 students choose Maths)

$= 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$

S: 2nd student picked Maths. $= \frac{1}{2}$

T: $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$T < S < R$.

I: symmetry by vertical line does not change over/underestimate by trapezium rule.

II: symmetry by horizontal line must change over/underestimate by trapezium rule.

13 A student approximates the integral $\int_a^b \sin^2 x \, \mathrm{d}x$ using the trapezium rule with 4 strips. The resulting approximation is an overestimate.

Which of the following is/are **necessarily** true?

I If the student approximates $\int_{-b}^{-a} \sin^2 x \, \mathrm{d}x$ in the same way, the result will be an overestimate.

II If the student approximates $\int_a^b \cos^2 x \, \mathrm{d}x$ in the same way, the result will be an underestimate.

A neither of them

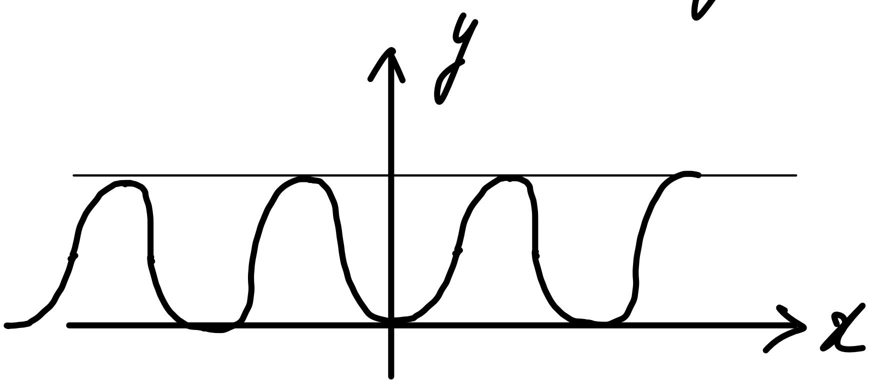
B I only

C II only

D I and II

I. $\sin^2 x = \sin^2(-x)$ thus even function.

(also checked graphically).



II. $\int_a^b \cos^2 x \, dx = \int_a^b (1 - \sin^2 x) \, dx$

let $f(x) = \sin^2 x$: Given $\int_a^b f(x) dx$ is overestimate,

flip by horizontal line $y = \frac{1}{2}$

$\int_a^b (1 - f(x)) \, dx$ is an underestimate.

* Don't obsess over the term "condition" just because they say condition does not mean it must be sufficient.

14 Consider the following statements about the polynomial $\mathrm{p}(x)$, where $a < b$:

I $\mathrm{p}(a) \le \mathrm{p}(b)$

~~II~~ $\mathrm{p}'(a) \le \mathrm{p}'(b)$

~~III~~ $\mathrm{p}''(a) \le \mathrm{p}''(b)$

Which of these statements is a **necessary** condition for $\mathrm{p}(x)$ to be increasing for $a \le x \le b$?

Definition of increasing function:

if $a<b$, then $p(a)<p(b)$

$\boxed{a<b} \rightarrow \boxed{p(a)<p(b)}$

hence necessary condition.

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

☆ simple substitution

cf) to be fair you don't need to substitute it & add new variables.

15 The numbers a, b and c are each greater than 1.

The following logarithms are all to the same base:

$$\log(ab^2c) = 7$$
$$\log(a^2bc^2) = 11$$
$$\log(a^2b^2c^3) = 15$$

What is this base?

A a

B b

C c

D It is possible to determine the base, but the base is not a, b or c.

E There is insufficient information given to determine the base.

$$\left.\begin{aligned} \log a + 2\log b + \log c &= 7 \\ 2\log a + \log b + 2\log c &= 11 \\ 2\log a + 2\log b + 3\log c &= 15. \end{aligned}\right\} \rightarrow \text{let } (\log a, \log b, \log c) = (A, B, C)$$

s.t $A, B, C \in (-\infty, \infty)$

$$\left.\begin{aligned} A + 2B + C &= 7 \\ 2A + B + 2C &= 11 \\ 2A + 2B + 3C &= 15 \end{aligned}\right\}. \quad \begin{aligned} &\cdots ① \\ &\cdots ② \\ &\cdots ③ \end{aligned}$$

② − 2×① :

$(2A + B + 2C) - 2\times(A + 2B + C)$

$= -3B = 11 - 14$

$\therefore B = 1$ $\quad \therefore$ base $= b$.

* you don't need to find the rest of A,B,C

cf) It's an inequality so the boundaries must be excluded → dotted line.

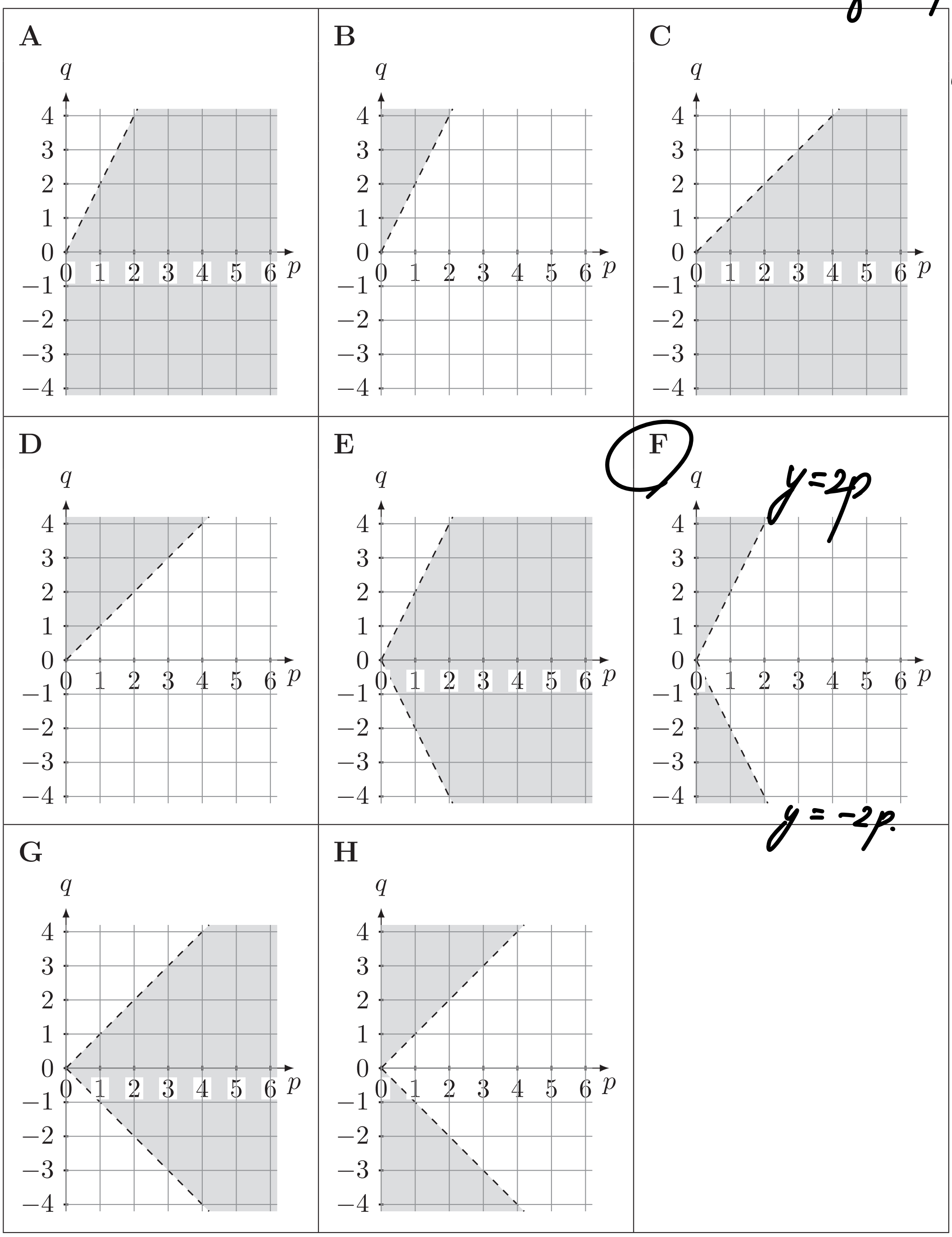
16 The graph of the quadratic

$$y = px^2 + qx + p$$

where $p > 0$, intersects the x-axis at two distinct points.) $D > 0$ $q^2 - 4p^2 > 0$

In which one of the following graphs does the **shaded** region show the complete set of possible values that p and q could take?

$q > 2p$ or $q < -2p$. $(\because p > 0)$



17 A multiple-choice test question offered the following four options relating to a certain statement:

B

A The statement is true **if and only if** $x > 1$

B The statement is true **if** $x > 1$

C The statement is true **if and only if** $x > 2$

D The statement is true **if** $x > 2$

Given that **exactly one** of these options was correct, which one was it?

A⊂B. } hence, if A is true then B is true.
C⊂D. " C " , D " .

Hence, A,C can't be true as they would contradict this bit.

i) Assume B is true.

→ If $x>1$, does not guarantee that $x>2$.
e.g) $x=1.5$.

ii) Assume D is true.

If $x>2$, x must be bigger than 1.

☆ Again typical example of two functions where one we know as fixed, and the rest we do not know of.

☆ In'equalities start from Equalities!

18 Consider the following inequality:

$$(*): \quad a|x| + 1 \leq |x-2|$$

where a is a real constant.

Which one of the following describes the complete set of values of a such that $(*)$ is **true for all real** x?

A $a \leq \frac{3}{2}$

B $a \leq 1$

C $a \leq \frac{1}{2}$

D $a \leq 0$

E $a \leq -\frac{1}{2}$

F $a \leq -1$

G $a \leq -\frac{3}{2}$

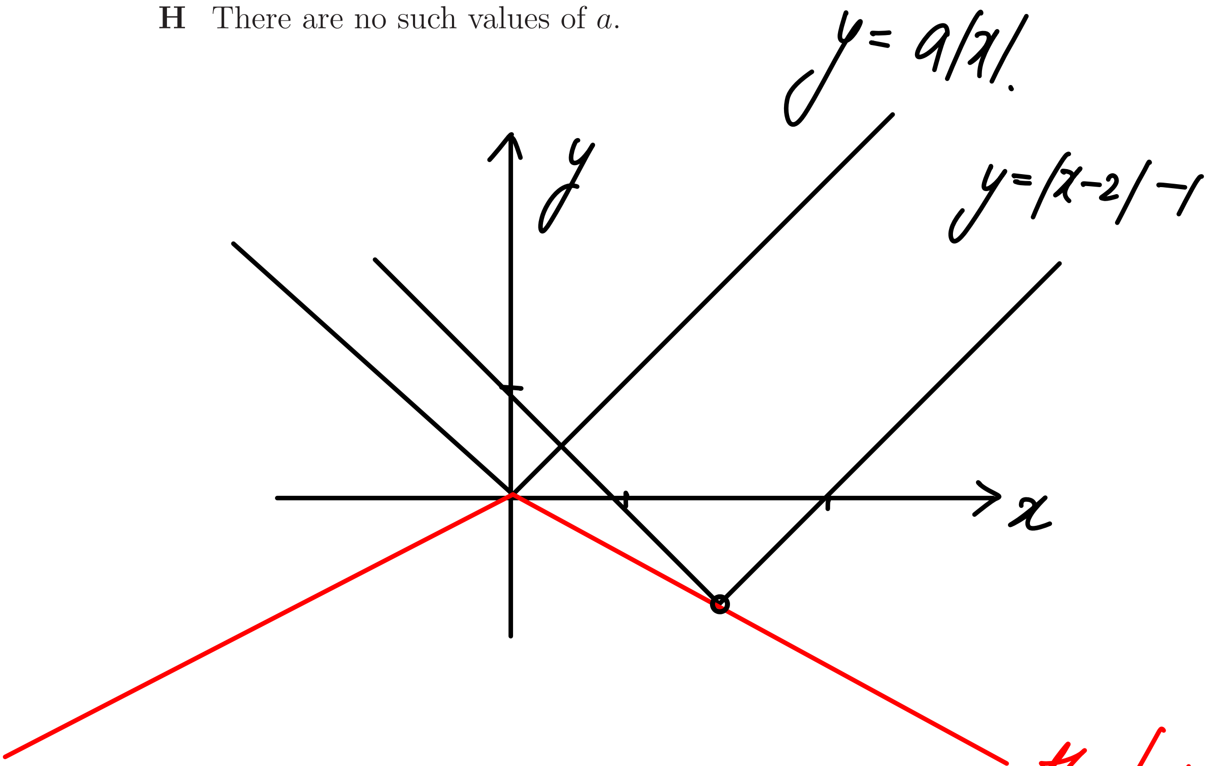
H There are no such values of a.

$a|x|+1 \leq |x-2|$

$\downarrow$

$a|x| \leq |x-2| - 1$

I did this process to simplify the function that we do not know of.



the boundaries s.t. $y = a|x|$ intersects with $y = |x-2| - 1$

for $\sqrt{a+b\sqrt{c}}$ formats, it's key to try to complete squares for whatever is in the surd ⊕

make $b=2$.

19 Find the value of the expression

$$\sqrt{8-4\sqrt{2}+1}+\sqrt{9-12\sqrt{2}+8}$$

$$\sqrt{9-2\sqrt{8}}+\sqrt{17-2\sqrt{72}}$$

$$\sqrt{(\sqrt{8}-\sqrt{1})^2}+\sqrt{(\sqrt{9}-\sqrt{8})^2}$$

$$=\sqrt{8}-\sqrt{1}+\sqrt{9}-\sqrt{8}$$

$$=-1+3=2$$

A $\sqrt{26-16\sqrt{2}}$

B $4\sqrt{2}-4$

C -2

D $4-4\sqrt{2}$

E 2

F $\sqrt{26}-4\sqrt{2}$

G 1

* Do the final substitution later on.

20 When the graph of the function $y = \mathrm{f}(x)$, defined on the real numbers, is reflected in the y-axis and then translated by 2 units in the negative x-direction, the result is the graph of the function $y = \mathrm{g}(x)$.

When the graph of the same function $y = \mathrm{f}(x)$ is translated by 2 units in the negative x-direction and then reflected in the y-axis, the result is the graph of the function $y = \mathrm{h}(x)$.

Which one of the following conditions on $y = \mathrm{f}(x)$ is **necessary and sufficient** for the functions $\mathrm{g}(x)$ and $\mathrm{h}(x)$ to be identical?

A $\mathrm{f}(x) = \mathrm{f}(x + 2)$ for all x

B $\mathrm{f}(x) = \mathrm{f}(x + 4)$ for all x

C $\mathrm{f}(x) = \mathrm{f}(x + 8)$ for all x

D $\mathrm{f}(x) = \mathrm{f}(-x)$ for all x

E $\mathrm{f}(x) = \mathrm{f}(2 - x)$ for all x

F $\mathrm{f}(x) = \mathrm{f}(4 - x)$ for all x

G $\mathrm{f}(x) = \mathrm{f}(8 - x)$ for all x

① $f(x, y) = 0$

$\downarrow$ y-axis reflection.

$f(-x, y) = 0.$

$\downarrow$ $x: -2$

$f(-(x+2), y) = 0.$

$\therefore g(x) = f(-x-2)$

② $f(x, y) = 0$

$\downarrow$ $x: -2$

$f(x+2, y) = 0.$

$\downarrow$ y axis reflection.

$f(-x+2, y) = 0.$

$h(x) = f(-x+2)$

cyclical by a length of 4.

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TMUA9.
Jamie Baek.
(백선후)