Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

TMUA9. Analysis.

PAPER 1 **D513/01**

2022 **75 minutes**

5581773094

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

This paper is the first of two papers.

This paper contains 20 questions. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

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1 How many real solutions are there to the equation

$$2\cos^4\theta - 5\cos^2\theta + 3 = 0$$

in the interval $0 \leq \theta \leq 2\pi$?

A 1

B 2

C 3

D 4

E 5

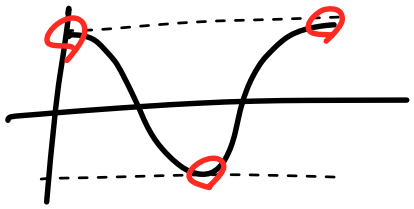
F 6

G 7

H 8

$(2\cos^2\theta - 3)(\cos^2\theta - 1) = 0$.

$\cos\theta = \pm 1$



2 Find the complete set of values of p for which the equation

$$x^2 - 2px + y^2 - 6y - p^2 + 8p + 9 = 0$$

describes a circle in the xy-plane.

A $p < -\frac{9}{4}$

B $0 < p < 4$

C $-1 < p < 9$

D $p < 0$ or $p > 4$

E $p < -1$ or $p > 9$

F all real values of p

$x^2 - 2px + p^2 + y^2 - 6y + 9$

$-2p^2 + 8p = 0.$

$(x-p)^2 + (y-3)^2 = 2p^2 - 8p$ ★

$2p(p-4) > 0.$

$p > 4$ or $p < 0$

3 Given the following statements about a function f

- $f''(x) = a$ for all x
- $f(0) = 1,\ f(1) = 2$
- $\int_0^1 f(x)\,dx = 1$

find the value of a.

A −6

B −3

C −2

D 2

E 3

F 6

$f''(x) = a$

$f'(x) = ax + b$

$f(x) = \frac{1}{2}ax^2 + bx + 1$

$f(1) = \frac{1}{2}a + b + 1 = 2.$

$\frac{1}{2}a + b = 1 \quad \cdots (1)$

$\left[\frac{1}{6}ax^3 + \frac{1}{2}bx^2 + x\right]_0^1$

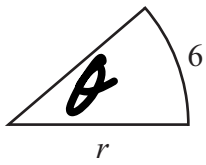
$= \frac{1}{6}a + \frac{1}{2}b + 1 = 1$

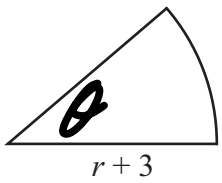
$a + 3b = 0 \quad \cdots (2)$

$-0.5b = 2. \quad b = -2 \quad a = 6.$

★ Every single circle is "similar." If angle of circular sector is identical, then the sectors are all similar (with ratio of similarity being ratio of radius)

4





These sectors of circles are similar.

The arc length of the smaller sector is 6.

The difference between the areas of the sectors is 21.

Find the positive difference between the perimeters of the sectors.

A 4.5

B 7

C 8

D 9

E 10.5

F 14

G 15

$2r + r\theta = r(\theta + 2)$

$(r+3)(\theta+2) - r(\theta+2)$

$\rightarrow 3(\theta+2) = 2 + 6 = 8$

$r\theta = 6$

$\frac{1}{2}(r+3)^2\theta - \frac{1}{2}r^2\theta = 21$

$\frac{1}{2}(6r + 9)\theta = 21$

$(2r+3)\theta = \frac{2}{3} \times 21 = 14$

$2r\theta = 12, \quad 3\theta = 2 \rightarrow \theta = \frac{2}{3}$

9. Speaking of Sequences.

5 The terms x_n of a sequence follow the rule

$$x_{n+1} = \frac{x_n + p}{x_n + q}$$

where p and q are real numbers.

Given that $x_1 = 3$, $x_2 = 5$, and $x_3 = 7$, find the value of x_4

A -5

B 5

C $\frac{51}{7}$

D $\frac{15}{2}$

E $\frac{23}{3}$

F 9

G 11

H 13

$5 = x_2 = \frac{x_1 + p}{x_1 + q} = \frac{3+p}{3+q}$

$7 = x_3 = \frac{x_2 + p}{x_2 + q} = \frac{5+p}{5+q}$.

$* \quad p+3 = 5(3+q)$

$p - 5q = 12 \quad \therefore p = 5q + 12$

$= -45 + 12$

$* \quad p+5 = 7(q+5)$ $\quad p = -33$

$p - 7q = 30$

$2q = -18. \quad q = -9.$

$x_4 = \frac{x_3 - 33}{x_3 - 9} = \frac{7-33}{7-9} = 13$

9. Recurrence Relations.

*its quite common for at least 1 question in paper 2 for TMUA to include some form of recursion. There are two basic attitudes for it.

① It's a sequence. Singular reaction circuit for sequences: jot them out.

② As... you jot them out, you must anticipate cycles. It's usually the case that the first terms are volatile thus don't adhere to the pattern. But as you write them out, it shows a very obvious pattern.

9. In the context of logarithms.

6 Given that

$$\int_{\log_2 5}^{\log_2 20} x \, dx = \log_2 M$$

what is the value of M?

A 4

B 15

C 16

D 20

E 25

F 100

G 10 000

$$\left[\frac{1}{2}x^2\right]_{\log_2 5}^{\log_2 20}$$

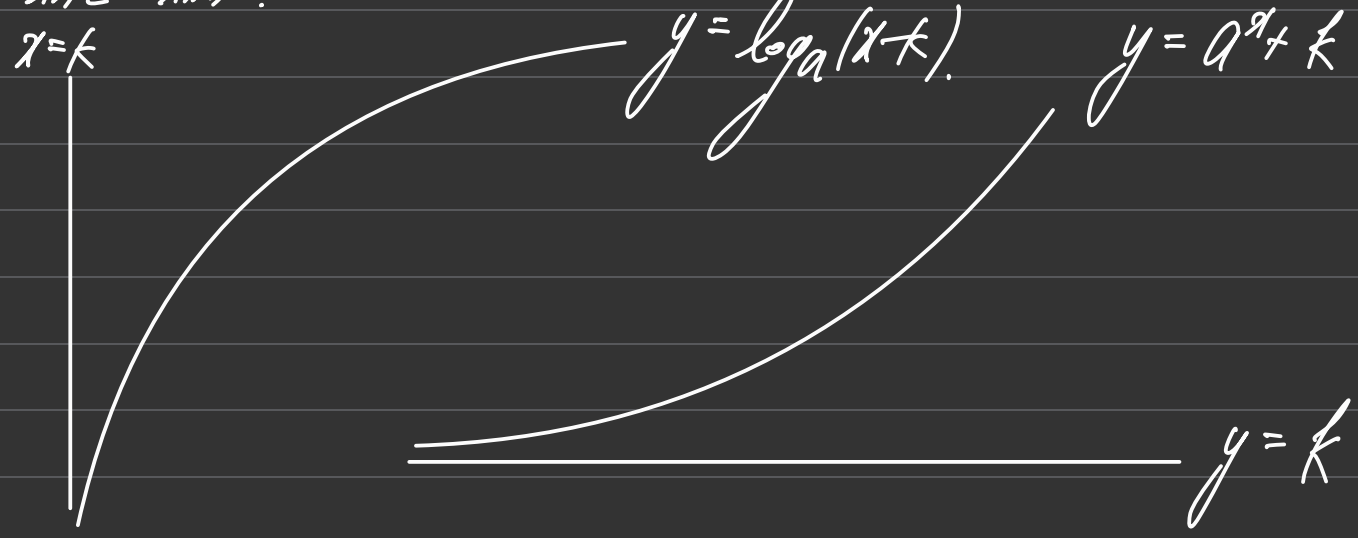
$$\frac{1}{2}\log_2 100 \times \log_2 4$$

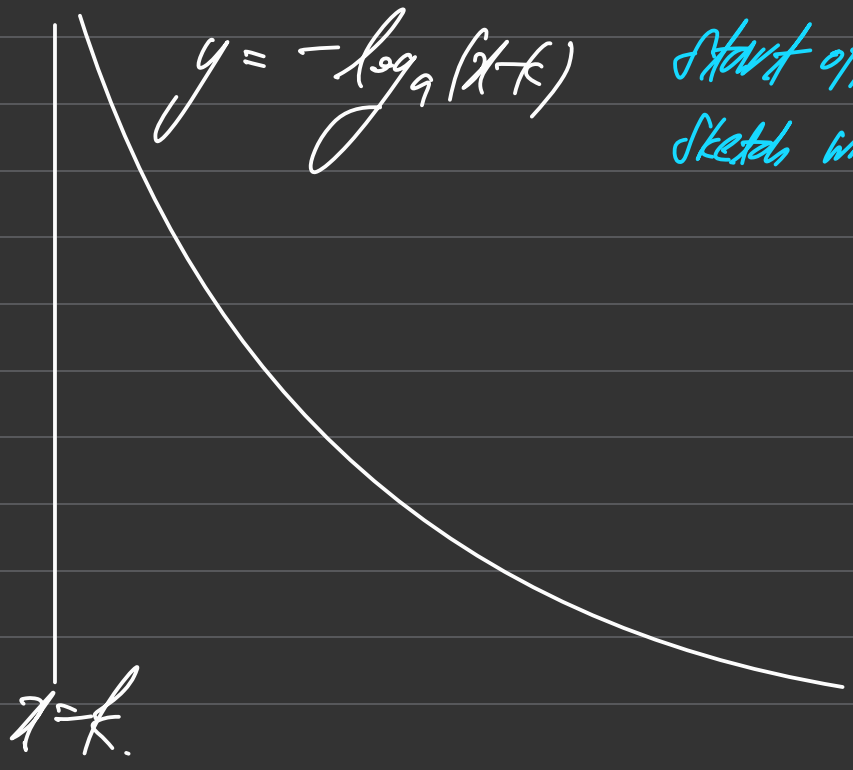
$$\log_2 100$$

9. Graphs for logarithmic & exponential.

free your mind. x-y axis are by no means needed.

all the time.





Don't automatically start off with x & y axis. Sketch what is needed.

7 Find the finite area enclosed between the line $y = 0$ and the curve $y = x^2 - 4|x| - 12$

A $\frac{128}{3}$

B $\frac{176}{3}$

C $\frac{256}{3}$

D 108

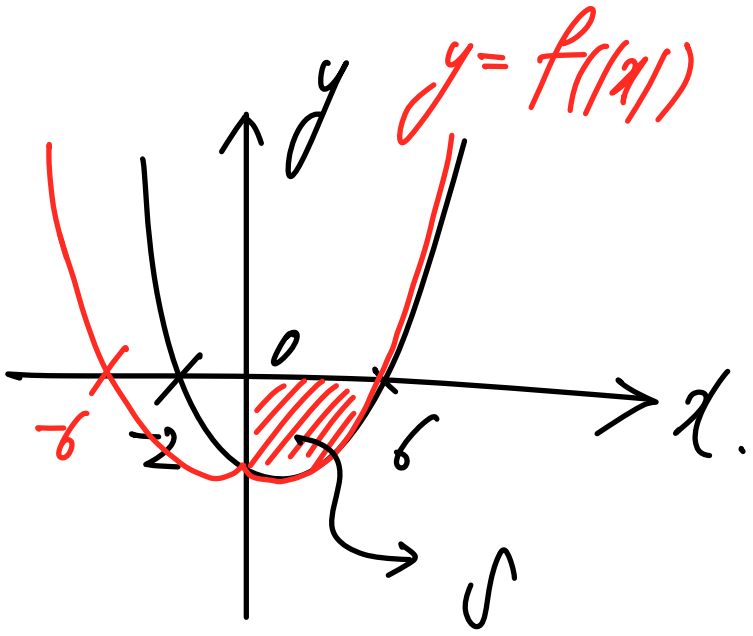
E 144

F 288

let $f(x) = x^2 - 4x - 12$

$= (x-6)(x+2)$

$f(|x|)$



$2S = 2\int_6^0 (x^2 - 4x - 12)\,dx$

$= 2\times\left[\frac{1}{3}x^3 - 2x^2 - 12x\right]_6^0$

$= 2\times\left(-\frac{1}{3}\times 6^3 + 2\times 6^2 + 12\times 6\right) = 144$

-72 $\quad = 144$

9. Generalisation.

8 A geometric sequence has first term a and common ratio r, where a and r are positive integers and r is greater than 1.

The sum of the first n terms of this sequence is denoted by S_n

It is given that the terms of the sequence satisfy

$$S_{30} - S_{20} = kS_{10}$$

for some positive integer k.

What is the smallest possible value of k?

A 2^{10}

B 2^{20}

C 2^{30}

D $\dfrac{2^{10}}{2^{10} - 1}$

E $2^{10}(2^{10} - 1)$

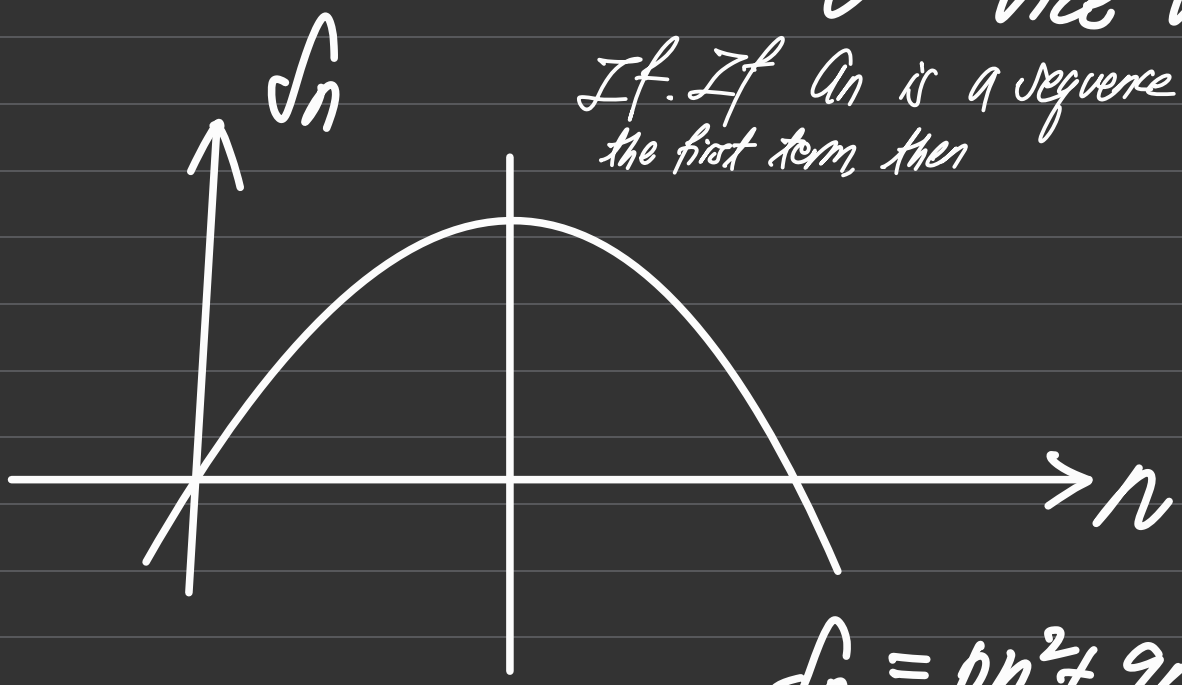
$a_{21} + a_{22} + \cdots + a_{30}$

$= k(a_1 + a_2 + \cdots + a_{10})$

$k = r^{20}$

9. Quantum leaps to finding a_n from S_n, vice versa.

If. If a_n is a sequence from the the first term, then



$$S_n = pn^2 + qn$$

$\rightarrow$ A polynomial in terms of n.
$\rightarrow$ Line symmetric about the vertical line.

$$\therefore a_n \cong S_n,$$
$$a_1 = S_1.$$

If $S_n = pn^2 + qn$, $a_n = 2pn + \star$.

$S_1 = p + q$ (Sum of coefficients), $a_1 = 2p + \star$.

If $a_n = a + d(n-1) = dn + C$ (C = Real Constant), then $S_n = \frac{1}{2}dn^2 + \star n$

$a_1 = S_1$, so $a = \frac{1}{2}d + C$.

9. why not sol(2 ?)

9 This question is about pairs of functions f and g that satisfy

$$f(x) - g(x) = 2\sin x$$

$$f(x)\,g(x) = \cos^2 x$$

for all real numbers x.

Across all solutions for $f(x)$, what is the minimum value that $f(x)$ attains for any x?

A $1 - \sqrt{2}$

B $-1 - \sqrt{2}$

C 0

D -1

E -2

F -3

G -4

maybe. $g(x) = f(x) - 2\sin x$

$f(x)\,g(x) = (f(x))^2 - 2\sin x\, f(x) = (1 - \sin^2 x)$

$(f(x))^2 - 2\sin x \times f(x) + (\sin x - 1)(\sin x + 1) = 0.$

$(f(x) - \sin x - 1)(f(x) - \sin x + 1) = 0.$

$f(x) = \sin x + 1.\ \sin x - 1.$

$\min(|f(x)|)$ at $\sin x = 1. \rightarrow 1 - 1 = 2.$

9. Sol 2.

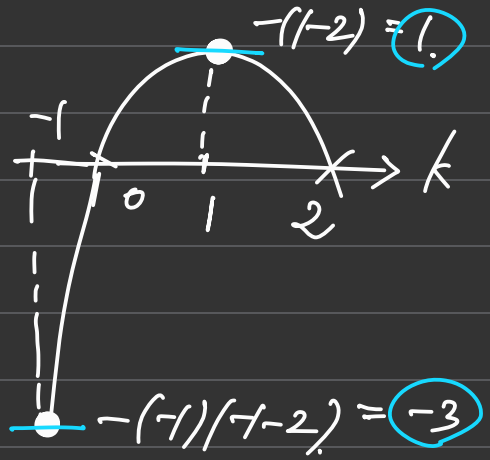
Sol 2) $f(x)g(x) + f(x) - g(x) - 1 = \cos x^2 + 2\sin x - 1$.

$= -\sin^2 x + 2\sin x$.

$(f(x)-1)(g(x)+1) = -\sin x(\sin x - 2)$.

What now?

RHS $= -k(k-2)$ s.t $k = \sin x$, & $k \in [-1, 1]$.



$-3 \le (f(x)-1)(g(x)+1) \le 1$

But it just goes back when

$-2 \le f(x) - g(x) \le 2$. ($\because f(x) - g(x) = 2\sin x$)

$0 \le f(x)g(x) \le 1$ ($\because f(x)g(x) = \cos^2 x$)

* Translation must be in the form of $f(x-\square)+\square$.
* Don't leave out translation parallel form. → to y-axis only can affect constant.

10 A sequence of translations is applied to the graph of $y = x^3$

Which of the following graphs could be the result of this sequence of translations?

I $y = x^3 - 3x^2 + 9x - 27$

II $y = x^3 - 9x^2 + 27x - 3$

III $y = 27x^3 - 9x^2 + x - 3$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

↳ if constant that is multiplied to $f(x)$ isn't ± 1, then does not count as translations UNLESS

$(x-3)^3$

$= x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3$

$= x^3 - 9x^3 + 27x - 27$.

$\Rightarrow x^3 - 9x^3 + 27x - 3$

$= (x-3)^3 - 24$.

for exponential / logarithmic functions.

9. Rare overlap of translations & multiplication (scaling)

£ Addition of log = Product

C

11 Evaluate

$$\sum_{n=1}^{100} \log_{10}\left(3^{1-n}\right)$$

A $-4950 \log_{10} 3$

B $4950 \log_{10} 3$

C $-5050 \log_{10} 3$

D $5050 \log_{10} 3$

E $1 - 4950 \log_{10} 3$

F $1 + 4950 \log_{10} 3$

G $1 - 5050 \log_{10} 3$

H $1 + 5050 \log_{10} 3$

$\log_{10} 3^0 + \cdots + \log_{10} 3^{-99}$

$= -(0+1+\cdots+99)\log_{10} 3.$

cf) Expression 차이.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_k + \cdots + a_{n-1} + a_n$$

$$\prod_{k=1}^{n} a_k = a_1 \times a_2 \times a_3 \times \cdots \times a_k \times \cdots \times a_{k-1} \times a_k$$

→ notation of cumulative product.

9. Definitive expression of a function

D

12 A family of quadratic curves is given by

$$y_k = 2\left(x - \frac{k}{2}\right)^2 + \frac{k^2}{2} + 4k + 3$$

where k is any real number and y_k is a function of x.

All these curves are sketched, and the point with the lowest y-coordinate among all the curves y_k is (a, b).

Find the value of $a + b$

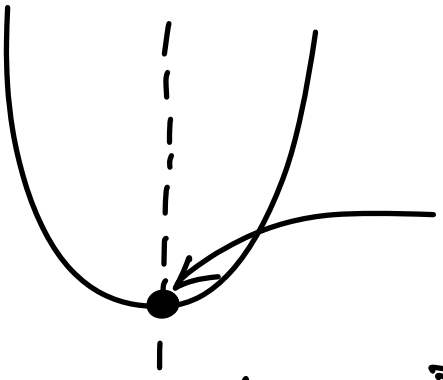
A –1

B –3

C –5

D –7

E –9



$\frac{1}{2}k^2+4k+3$

$\frac{1}{2}(k^2+8k+16)-5$

$\frac{1}{2}(k+4)^2-5$

$k=-4$

$x=\frac{k}{2}$

$=-2$

$a=-2$

$b=-5$

± multiply term for factor.

9. Cauchy Schwartz.

13 Given that

$$\left(a^3+\frac{2}{b^3}\right)\left(\frac{2}{a^3}-b^3\right)=\sqrt{2}$$

where a and b are real numbers, what is the least value of ab?

A $-\sqrt{2}$

B $\sqrt{2}$

C $-2\sqrt{2}$

D $2\sqrt{2}$

E $-\frac{\sqrt{2}}{2}$

F $\frac{\sqrt{2}}{2}$

G $-2^{\frac{1}{6}}$

H $2^{\frac{1}{6}}$

$$2-a^3b^3+\frac{4}{a^3b^3}-2=\sqrt{2}$$

$$\frac{4}{\star}-\star=\sqrt{2}$$

$$-\star^2+4=\sqrt{2}\star$$

$$\star^2+\sqrt{2}\star-4=0$$

$$(\star+2\sqrt{2})(\star-\sqrt{2})=0.$$

$$\star=-2\sqrt{2},\ \sqrt{2}$$

$$a^3b^3=-2^{\frac{3}{2}},\ 2^{\frac{1}{2}}$$

$$ab=-2^{\frac{1}{2}},\ 2^{\frac{1}{6}}$$

$\angle$ max(S) = for fixed base = maximum height

14 A circle has centre O and radius 6.

P, Q and R are points on the circumference with angle $POQ \geq \frac{\pi}{2}$

The area of the triangle POQ is $9\sqrt{3}$

What is the greatest possible area of triangle PRQ?

A $18 + 9\sqrt{3}$

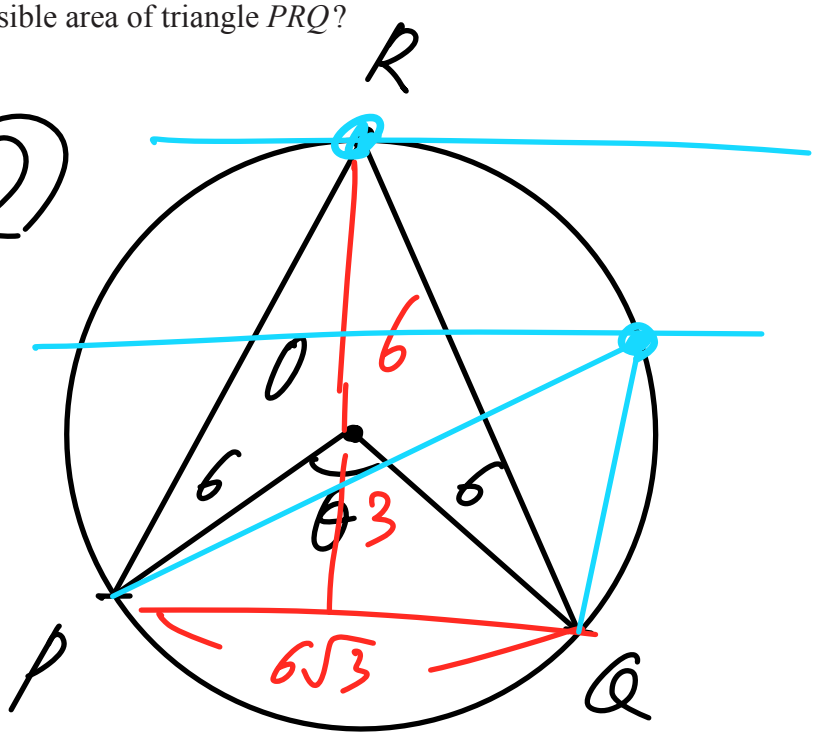
B $18\sqrt{3}$

C $27 + 9\sqrt{3}$

D $27\sqrt{3}$

E $36 + 9\sqrt{3}$

F $36\sqrt{3}$



D

$\frac{1}{2} \cdot 6^2 \cdot \sin\theta = 9\sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60^\circ, 120^\circ$

9. Visual Generalisation

$6\sqrt{3} \times 9 \times \frac{1}{2} =$

*Again, sketching the y axis was not needed.

15 A rectangle is drawn in the region enclosed by the curves p and q, where

$$p(x) = 8 - 2x^2$$

$$q(x) = x^2 - 2$$

such that the sides of the rectangle are parallel to the x- and y-axes.

What is the maximum possible area of the rectangle?

A $\frac{26}{9}$

B $\frac{52}{9}$

C $\frac{4\sqrt{6}}{3}$

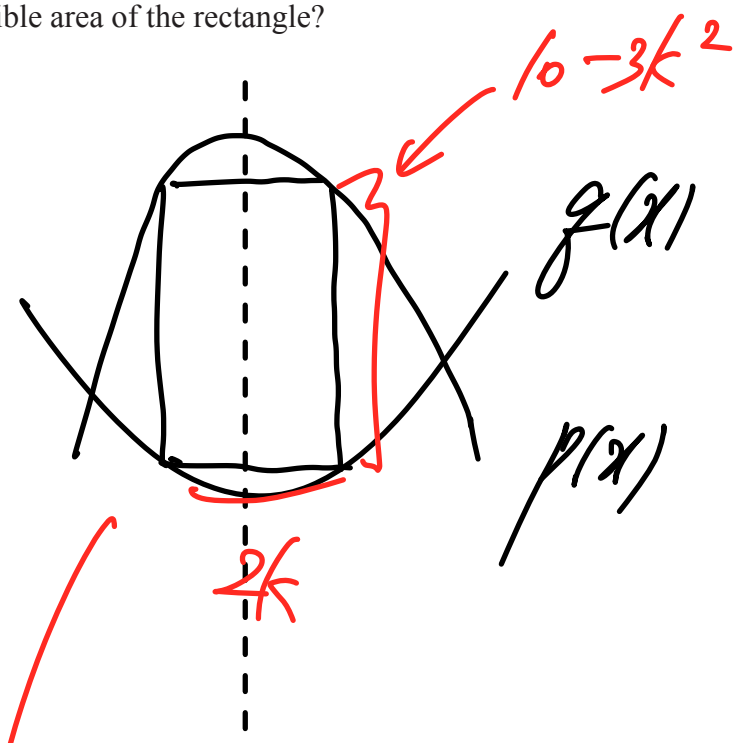
D $\frac{8\sqrt{6}}{3}$

E $4\sqrt{2}$

F $8\sqrt{2}$

G $\frac{20\sqrt{10}}{9}$

H $\frac{40\sqrt{10}}{9}$



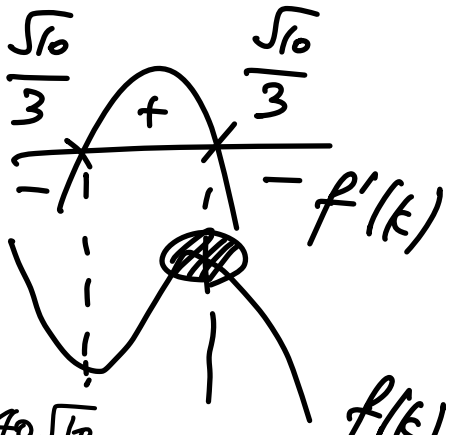
$8 - 2k^2 - k^2 + 2$

$= 10 - 3k^2$

$2k(10 - 3k^2)$

$f(k) = -6k^3 + 20k$

$f'(k) = -18k^2 + 20$



$f\left(\frac{\sqrt{10}}{3}\right) = -2k(3k^2 - 10)$

$= \frac{-2\sqrt{10}}{3}\left(\frac{-20}{3}\right) = \frac{40\sqrt{10}}{9}$

* $\sin^2\theta + \cos^2\theta = 1$. & Diff of Squares.

16 The solutions to $7x^4 - 6x^2 + 1 = 0$ are $\pm\cos\theta$ and $\pm\cos\beta$.

Which one of the following equations has solutions $\pm\sin\theta$ and $\pm\sin\beta$?

A $7x^4 - 8x^2 - 5 = 0$

B $7x^4 - 8x^2 + 2 = 0$

C $7x^4 - 6x^2 - 2 = 0$

D $7x^4 - 6x^2 + 1 = 0$

E $7x^4 + 6x^2 - 1 = 0$

F $7x^4 + 6x^2 + 5 = 0$

$7(x^2 - \cos^2\theta)(x^2 - \cos^2\beta) = 0.$

$7\left(x^4 - (\cos^2\theta + \cos^2\beta)x^2 + \cos^2\theta\cos^2\beta\right) = 0$

$7x^4 - \underbrace{7(\cos^2\theta + \cos^2\beta)}_{=-6.}x^2 + \underbrace{7\cos^2\theta\cos^2\beta}_{=1.} = 0.$

$\cos^2\theta + \cos^2\beta = \frac{6}{7}$

$\cos^2\theta \times \cos^2\beta = \frac{1}{7}$

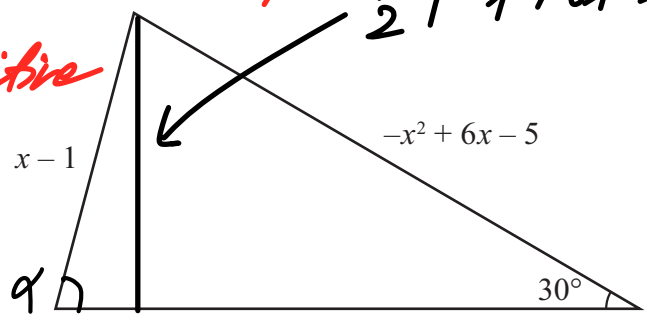
$(x^2 - \sin^2\theta)(x^2 - \sin^2\beta) = 0.$

$-(\sin^2\theta + \sin^2\beta) = (\cos^2\theta - 1) + (\cos^2\beta - 1) = \frac{6}{7} - 2 = -\frac{8}{7}.$

$\sin^2\theta\sin^2\beta = (1 - \cos^2\theta)(1 - \cos^2\beta) = 1 - \frac{6}{7} + \frac{1}{7} = \frac{2}{7}$

* Basic conditions for triangle: Shorter sides.
* longest side is shorter than sum of
* all sides are positive

17 ★



Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A $1 < x < 3$

B $1 < x < 4$

C $1 < x < 5$

D $3 < x < 4$

E $3 < x < 5$

F $4 < x < 5$

9. triangle conditions.

$x+1 > \frac{1}{2}(-x^2+6x-5)$

$2x+2 > -x^2+6x-5$

$x^2-6x+5+2x-2 > 0$

$x^2-4x+3 > 0$

$x > 3$ or $x < 1$ ··· ①

Don't eliminate what's obvious. As each sides are positive: $x-1 > 0$, $-x^2+6x-5 > 0$

Magnitude comparison must be explicit.

* $x-1 < -x^2+6x-5$

$(x-1) < (x-1)(5-x)$

$(x-1)(1-5+x) < 0$

$(x-1)(x-4) > 0$

$\therefore x < 4$ or $x > 1$ ··· ③

↓ $(x+1)(x-5) < 0$

$1 < x < 5$ ··· ②

① ∩ ② ∩ ③ → $3 < x < 4$

learn how to sketch the polynomials (sketch end behaviour/ pay attention to intercepts)

18 It is given that

$$f(x) = x^2(x-1)^2(x-2)$$

$$g(x) = -p(x-q)^2(x-r)^2$$

where p, q and r are positive and $q < r$

p is positive & two positive roots.

Find the set of values of q and r that guarantees the greatest number of distinct real solutions of the equation $f(x) = g(x)$ for all p.

A $q < 1$ and $r < 1$

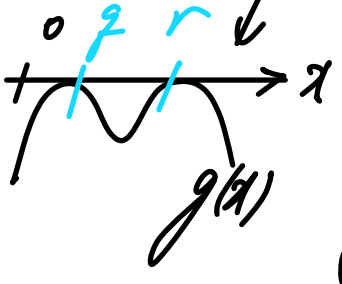
B $q < 1$ and $1 < r < 2$

C $q < 1$ and $r > 2$

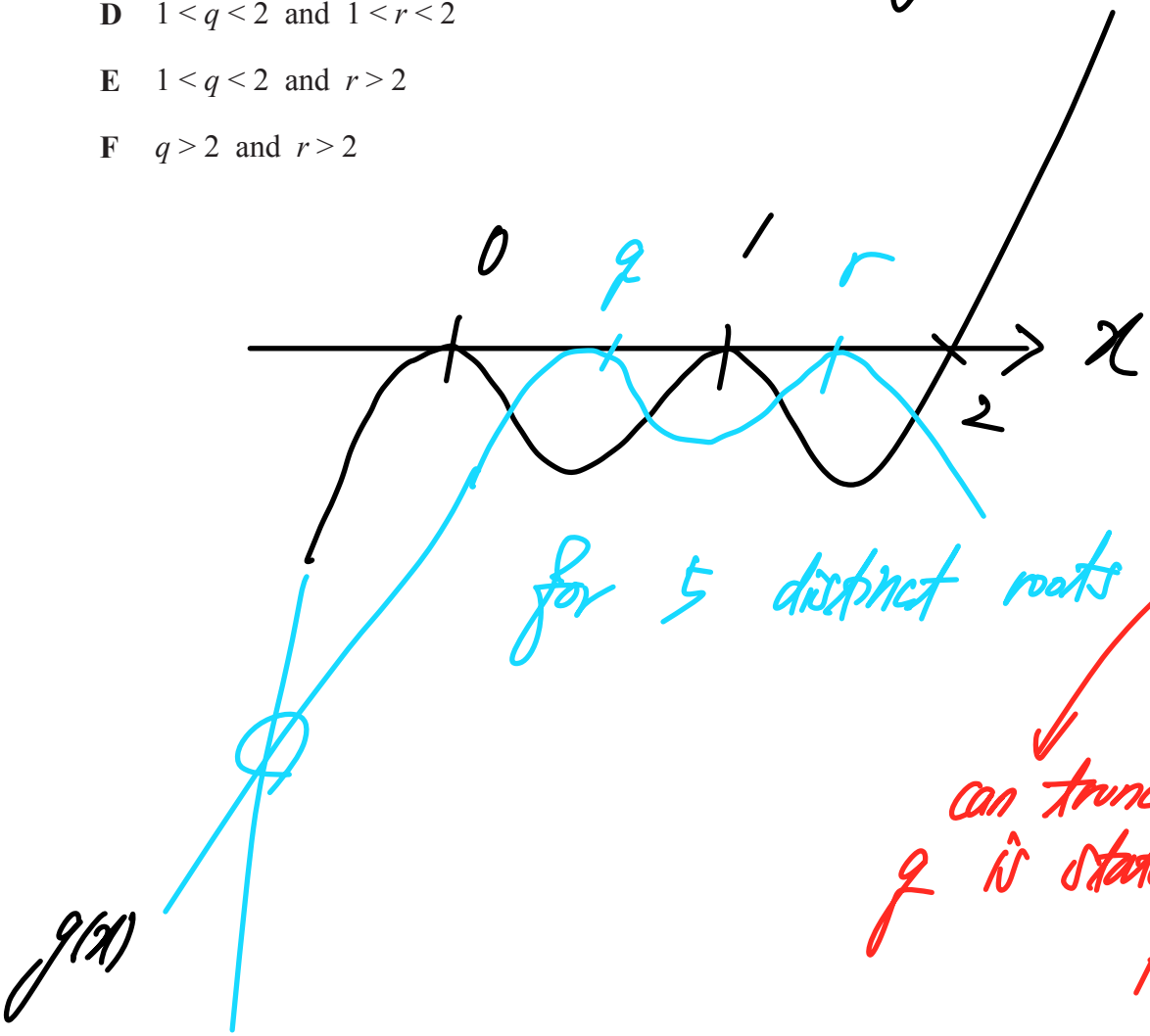
D $1 < q < 2$ and $1 < r < 2$

E $1 < q < 2$ and $r > 2$

F $q > 2$ and $r > 2$



y = f(x)



for 5 distinct roots $\begin{cases} 0<q<1 \\ 1<r<2. \end{cases}$

can truncate as q is stated as positive,,

☆ Excluding is super common in calculating probability. (complementary set (truth set of negated proposition))

19 Circle C_1 is defined as $x^2 + y^2 = 25$

A second circle C_2 has radius 4 and centre (a, b) where

$$-2 \le a \le 2 \quad \text{and} \quad -3 \le b \le 3$$

If the centre of C_2 is equally likely to be located anywhere within the given range, what is the probability that C_2 intersects C_1?

A $\frac{1}{25}$

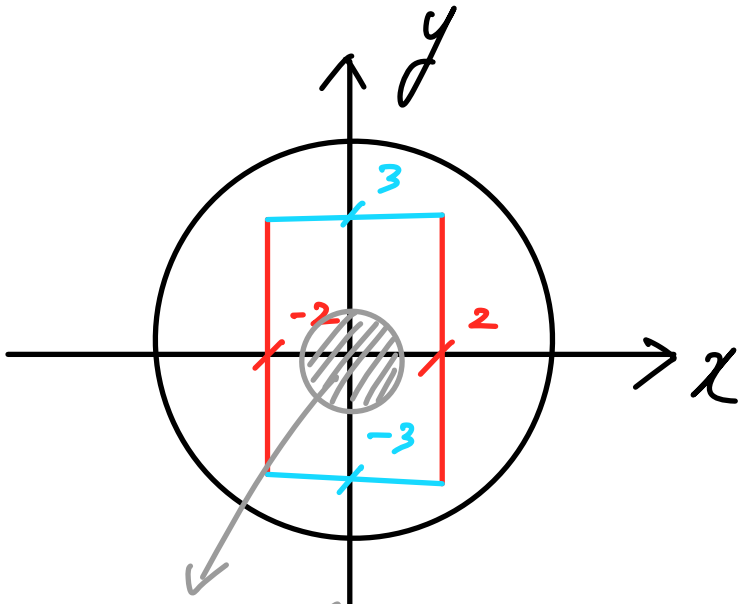
B $\frac{9}{25}$

C $\frac{16}{25}$

D $\frac{6-\pi}{6}$

E $\frac{16-\pi}{24}$

F $\frac{24-\pi}{24}$



if the centre of C_2 is within the grey-shaded area, what ends up happening is that they have no intersections.

(When C_2's centre is at the circumference of this unit circle $x^2+y^2=1$, the C_2 is inscribed to C_1.

→ Total = $(2-(-2)) \times (3-(-3)) = 4 \times 6 = 24$

Excluding $1^2\pi = \pi$.

$1 - \frac{\pi}{24} \Rightarrow \frac{24-\pi}{24}$

20 n is the number of points of intersection of the graphs

$$y = |x^2 - a^2| \text{ and } y = a^2|x - 1|$$

where a is a real number.

What is the smallest value of n that is **not** possible?

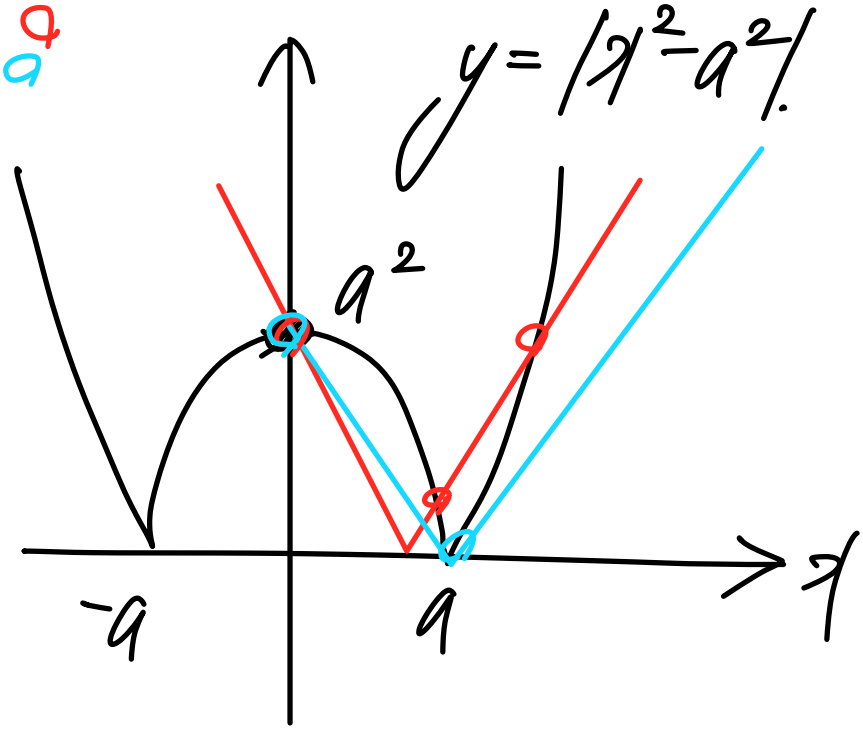
A $n = 1$

B $n = 2$

C $n = 3$

D $n = 4$

E $n = 5$



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