Cambridge Assessment Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

TMUA9. Model Answer.

PAPER 1 **D513/01**

2022 **75 minutes**

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

This paper is the first of two papers.

This paper contains 20 questions. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

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TMUA 9.

Jamie Baek

1 How many real solutions are there to the equation

$$2\cos^4\theta - 5\cos^2\theta + 3 = 0$$

in the interval $0 \le \theta \le 2\pi$?

A 1

B 2

C 3

D 4

E 5

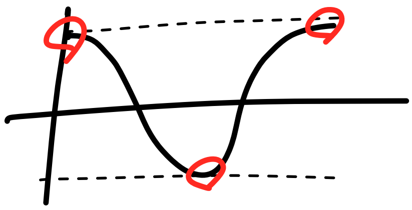
F 6

G 7

H 8

$(2\cos^2\theta - 3)(\cos^2\theta - 1) = 0.$

$\cos\theta = \pm 1$



2 Find the complete set of values of p for which the equation

$$x^2 - 2px + y^2 - 6y - p^2 + 8p + 9 = 0$$

describes a circle in the xy-plane.

A $p < -\frac{9}{4}$

B $0 < p < 4$

C $-1 < p < 9$

D $p < 0$ or $p > 4$

E $p < -1$ or $p > 9$

F all real values of p

$x^2 - 2px + p^2 + y^2 - 6y + 9$

$-2p^2 + 8p = 0.$

$$(x-p)^2 + (y-3)^2 = \underline{2p^2 - 8p}$$

$2p(p-4) > 0.$

$p > 4.\ p < 0$

3 Given the following statements about a function f

- $f''(x) = a$ for all x
- $f(0) = 1, \; f(1) = 2$
- $\int_0^1 f(x)\,dx = 1$

find the value of a.

A –6

B –3

C –2

D 2

E 3

F 6

$$f''(x) = a$$

$$f'(x) = ax + b$$

$$f(x) = \frac{1}{2}ax^2 + bx + 1$$

$$f(1) = \frac{1}{2}a + b + 1 = 2.$$

$$\frac{1}{2}a + b = 1 \quad \cdots ①$$

$$\left[\frac{1}{6}ax^3 + \frac{1}{2}bx^2 + x\right]_0^1$$

$$= \frac{1}{6}a + \frac{1}{2}b + 1 = 1$$

$$a + 3b = 0. \quad \cdots ②$$

$$-0.5b = 1. \quad b = -2 \quad a = 6.$$

* Every single circle is "similar." If angle of circular sector is identical, then the sectors are all similar (with ratio of similarity being ratio of radius)

4

These sectors of circles are similar.

The arc length of the smaller sector is 6.

The difference between the areas of the sectors is 21.

Find the positive difference between the perimeters of the sectors.

A 4.5

B 7

C 8

D 9

E 10.5

F 14

G 15

$$2r + r\theta = r(\theta+2)$$

$$(r+3)(\theta+2) - r(\theta+2)$$

$$\rightarrow 3(\theta+2) = 2+6 = 8$$

$$r\theta = 6$$

$$\frac{1}{2}(r+3)^2\theta - \frac{1}{2}r^2\theta = 21$$

$$\frac{1}{2}(6r+9)\theta = 21$$

$$(2r+3)\theta = \frac{2}{\cancel{3}} \times \cancel{21}^{7} = 14$$

$$2r\theta = 12, \quad 3\theta = 2 \rightarrow \theta = \frac{2}{3}$$

9. Speaking of Sequences.

5 The terms x_n of a sequence follow the rule

$$x_{n+1} = \frac{x_n + p}{x_n + q}$$

where p and q are real numbers.

Given that $x_1 = 3$, $x_2 = 5$, and $x_3 = 7$, find the value of x_4

A -5

B 5

C $\frac{51}{7}$

D $\frac{15}{2}$

E $\frac{23}{3}$

F 9

G 11

H 13

$$5 = x_2 = \frac{x_1+p}{x_1+q} = \frac{3+p}{3+q}$$

$$7 = x_3 = \frac{x_2+p}{x_2+q} = \frac{5+p}{5+q}.$$

$* \; p+3 = 5(3+q)$

$\underline{p-5q=12}$: $p = 5q+12$

$= -45+12$

$\underline{p=-33}$

$* \; p+5 = 7(q+5)$

$\underline{p-7q = 30}$

$2q = -18. \quad \underline{q=-9.}$

$$x_4 = \frac{x_3-33}{x_3-9} = \frac{7-33}{7-9} = 13$$

9. In the context of logarithms.

6 Given that

$$\int_{\log_2 5}^{\log_2 20} x \, dx = \log_2 M$$

what is the value of M?

A 4

B 15

C 16

D 20

E 25

F 100

G 10 000

$$\left[\frac{1}{2}x^2\right]_{\log_2 5}^{\log_2 20}$$

$$\cancel{\frac{1}{2}} \log_2 100 \times \cancel{\log_2 4}$$

$$\log_2 100$$

7 Find the finite area enclosed between the line $y = 0$ and the curve $y = x^2 - 4|x| - 12$

A $\frac{128}{3}$

B $\frac{176}{3}$

C $\frac{256}{3}$

D 108

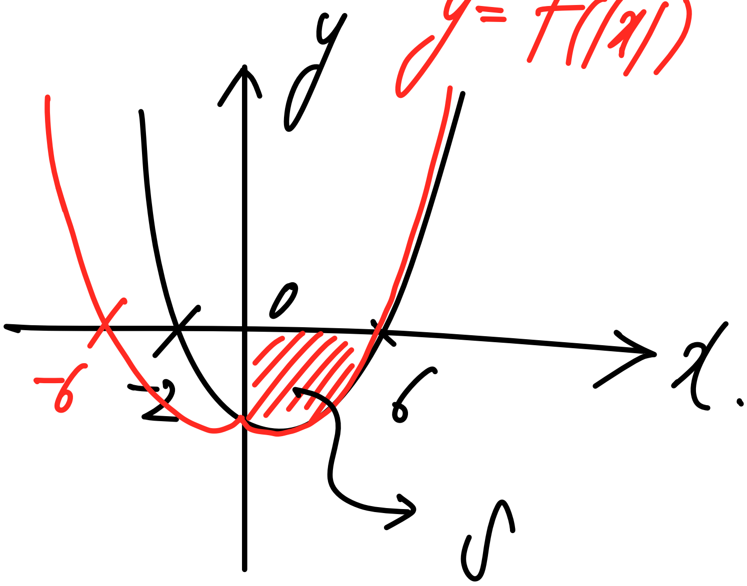
E 144

F 288

let $f(x) = x^2 - 4x - 12$

$= (x-6)(x+2)$

$f(|x|)$



$$2S = 2\int_6^0 (x^2 - 4x - 12)\,dx.$$

$$= 2\times\left[\frac{1}{3}x^3 - 2x^2 - 12x\right]_6^0$$

$$= 2\times\left(\underbrace{-\frac{1}{3}\times 6^3}_{-72.} \underbrace{+2\times 6^2 + 12\times 6}_{=144}\right) = 144$$

9. Generalization.

8 A geometric sequence has first term a and common ratio r, where a and r are positive integers and r is greater than 1.

The sum of the first n terms of this sequence is denoted by S_n

It is given that the terms of the sequence satisfy

$$S_{30} - S_{20} = kS_{10}$$

for some positive integer k.

What is the smallest possible value of k?

A 2^{10}

B 2^{20}

C 2^{30}

D $\dfrac{2^{10}}{2^{10}-1}$

E $2^{10}(2^{10}-1)$

$a_{21} + a_{22} + \cdots + a_{30}$

$= k(a_1 + a_2 + \cdots + a_{10})$.

$k = r^{20}$

9. why not sol 2 ?)

9 This question is about pairs of functions f and g that satisfy

$$f(x) - g(x) = 2\sin x$$

$$f(x)\,g(x) = \cos^2 x$$

for all real numbers x. ✓

Across all solutions for $f(x)$, what is the minimum value that $f(x)$ attains for any x?

A $1 - \sqrt{2}$

B $-1 - \sqrt{2}$

C 0

D -1

E -2

F -3

G -4

maybe.

$g(x) = f(x) - 2\sin x$

$f(x)g(x) = (f(x))^2 - 2\sin x\, f(x) = (1 - \sin^2 x)$

$(f(x))^2 - 2\sin x \times f(x) + (\sin x - 1)(\sin x + 1) = 0.$

$(f(x) - \sin x - 1)(f(x) - \sin x + 1) = 0.$

$f(x) = \sin x + 1.\ \sin x - 1.$

$\min(f(x))$ at $\sin x = -1. \rightarrow -1 - 1 = -2.$

* Translation must be in the form of $f(x-\star)+\square$.

* Don't leave out translation parallel form. → to y-axis only can affect constant.

10 A sequence of translations is applied to the graph of $y = x^3$

Which of the following graphs could be the result of this sequence of translations?

~~I~~ $y = x^3 - 3x^2 + 9x - 27$

II $y = x^3 - 9x^2 + 27x - 3$

~~III~~ $y = 27x^3 - 9x^2 + x - 3$

→ if constant that is multiplied to f(x) isn't ±1, then does not count as translations UNLESS for exponential / logarithmic functions.

$(x-3)^3$

$= x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3$

$= x^3 - 9x^2 + 27x - 27$

$\Rightarrow x^3 - 9x^2 + 27x - 3$

$= (x-3)^3 - 24$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

9. Rare overlap of translations & multiplication (scaling)

£ Addition of log = Product

11 Evaluate

$$\sum_{n=1}^{100} \log_{10}\left(3^{1-n}\right)$$

A $-4950 \log_{10} 3$

B $4950 \log_{10} 3$

C $-5050 \log_{10} 3$

D $5050 \log_{10} 3$

E $1 - 4950 \log_{10} 3$

F $1 + 4950 \log_{10} 3$

G $1 - 5050 \log_{10} 3$

H $1 + 5050 \log_{10} 3$

$\log_{10} 3^{0} + \cdots + \log_{10} 3^{-99}$

$-(0+1+\cdots+99)\log_{10} 3.$

cf) Expression tip.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_k + \cdots + a_{n-1} + a_n$$

$$\prod_{k=1}^{n} a_k = a_1 \times a_2 \times a_3 \times \cdots \times a_k \times \cdots \times a_{k-1} \times a_k$$

notation of cumulative product.

9. Definitive expression of a function

12 A family of quadratic curves is given by

D

$$y_k = 2\left(x - \frac{k}{2}\right)^2 + \frac{k^2}{2} + 4k + 3$$

where k is any real number and y_k is a function of x.

All these curves are sketched, and the point with the lowest y-coordinate among all the curves y_k is (a, b).

Find the value of $a + b$

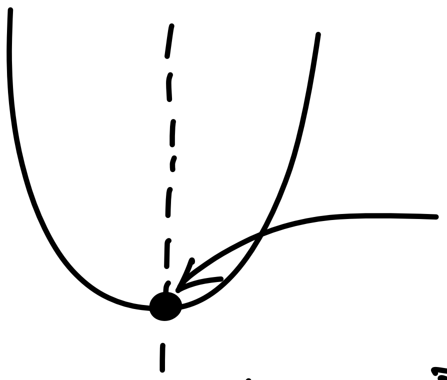
A -1

B -3

C -5

D -7

E -9



$\frac{1}{2}k^2+4k+3$

$\frac{1}{2}(k^2+8k+16)-5$

$\frac{1}{2}(k+4)^2-5$

$k=-4$

$x=\frac{k}{2}$

$=-2$

$a=-2$

$b=-5$

£ multiply term for factor.

9. Cauchy Schwartz.

13 Given that

A

$$\left(a^3 + \frac{2}{b^3}\right)\left(\frac{2}{a^3} - b^3\right) = \sqrt{2}$$

where a and b are real numbers, what is the least value of ab?

A $-\sqrt{2}$

B $\sqrt{2}$

C $-2\sqrt{2}$

D $2\sqrt{2}$

E $-\frac{\sqrt{2}}{2}$

F $\frac{\sqrt{2}}{2}$

G $-2^{\frac{1}{6}}$

H $2^{\frac{1}{6}}$

$$\cancel{2} - a^3b^3 + \frac{4}{\underbrace{a^3b^3}_{\star}} \cancel{-2} = \sqrt{2}$$

$$-\star + \frac{4}{\star} = \sqrt{2}$$

$$-\star^2 + 4 = \sqrt{2}\star$$

$$\star^2 + \sqrt{2}\star - 4 = 0$$

$$(\star + 2\sqrt{2})(\star - \sqrt{2}) = 0.$$

$$\star = -2\sqrt{2}, \sqrt{2}$$

$$a^3b^3 = -2^{\frac{3}{2}}, 2^{\frac{1}{2}}$$

$$ab = -2^{\frac{1}{2}}, 2^{\frac{1}{6}}$$

£max(S) = for fixed base = maximum height

14 A circle has centre O and radius 6.

P, Q and R are points on the circumference with angle $POQ \geq \frac{\pi}{2}$

The area of the triangle POQ is $9\sqrt{3}$

What is the greatest possible area of triangle PRQ?

A $18 + 9\sqrt{3}$

B $18\sqrt{3}$

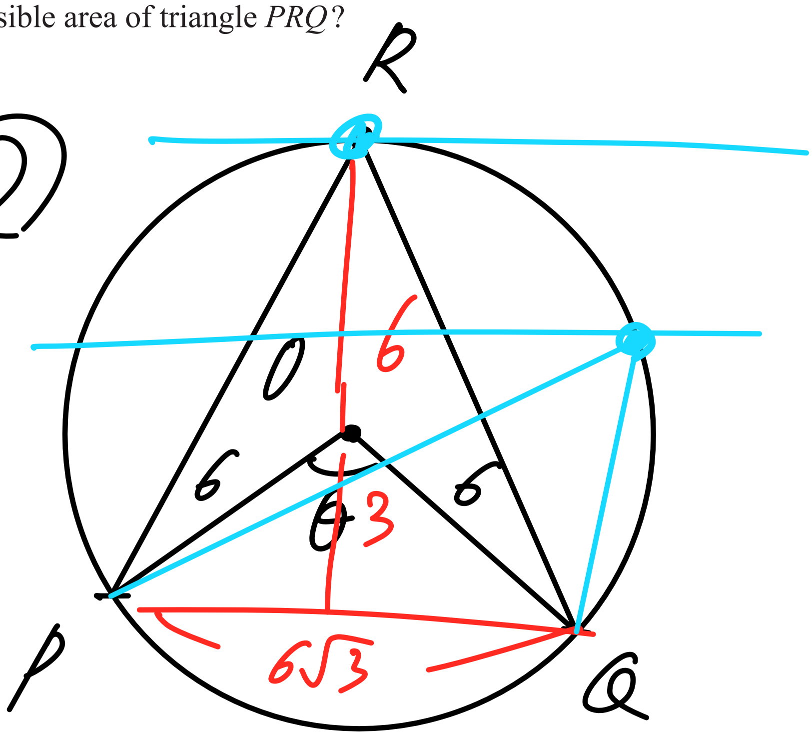
C $27 + 9\sqrt{3}$

D $27\sqrt{3}$

E $36 + 9\sqrt{3}$

F $36\sqrt{3}$

D



$\frac{1}{2} \cdot 6^2 \cdot \sin\theta = 9\sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60°; 120°$

$6\sqrt{3} \times 9 \times \frac{1}{2} =$

9. Visual Generalisation.

Again, sketching x & y axis was not needed.

15 A rectangle is drawn in the region enclosed by the curves p and q, where

$$p(x) = 8 - 2x^2$$

$$q(x) = x^2 - 2$$

such that the sides of the rectangle are parallel to the x- and y-axes.

What is the maximum possible area of the rectangle?

A $\frac{26}{9}$

B $\frac{52}{9}$

C $\frac{4\sqrt{6}}{3}$

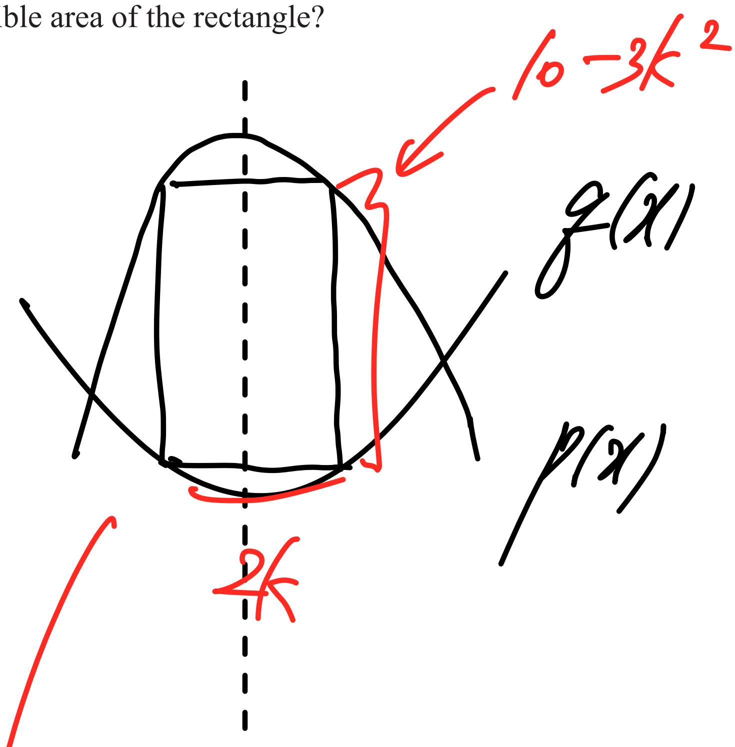
D $\frac{8\sqrt{6}}{3}$

E $4\sqrt{2}$

F $8\sqrt{2}$

G $\frac{20\sqrt{10}}{9}$

H $\frac{40\sqrt{10}}{9}$



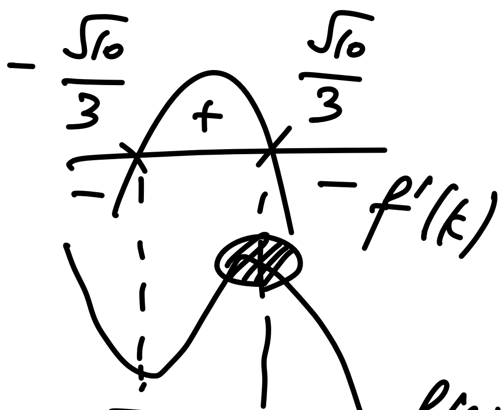
$8 - 2k^2 - k^2 + 2$

$= 10 - 3k^2$

$2k(10 - 3k^2)$

$f(k) = -6k^3 + 20k$

$f'(k) = -18k^2 + 20$



$f\left(\frac{\sqrt{10}}{3}\right) = -2k(3k^2 - 10)$

$= \frac{-2\sqrt{10}}{3}\left(\frac{-20}{3}\right) = \frac{40\sqrt{10}}{9}$

$\sin^2\theta + \cos^2\theta = 1$. & Diff of squares.

16 The solutions to $7x^4 - 6x^2 + 1 = 0$ are $\pm\cos\theta$ and $\pm\cos\beta$.

Which one of the following equations has solutions $\pm\sin\theta$ and $\pm\sin\beta$?

A $7x^4 - 8x^2 - 5 = 0$

B $7x^4 - 8x^2 + 2 = 0$

C $7x^4 - 6x^2 - 2 = 0$

D $7x^4 - 6x^2 + 1 = 0$

E $7x^4 + 6x^2 - 1 = 0$

F $7x^4 + 6x^2 + 5 = 0$

$$7(x^2 - \cos^2\theta)(x^2 - \cos^2\beta) = 0.$$

$$7\left(x^4 - (\cos^2\theta + \cos^2\beta)x^2 + \cos^2\theta\cos^2\beta\right) = 0$$

$$7x^4 \underbrace{- 7(\cos^2\theta + \cos^2\beta)}_{= -6.} x^2 + \underbrace{7\cos^2\theta\cos^2\beta}_{= 1.} = 0.$$

$$\begin{cases} \cos^2\theta + \cos^2\beta = \frac{6}{7} \\ \cos^2\theta \times \cos^2\beta = \frac{1}{7} \end{cases}$$

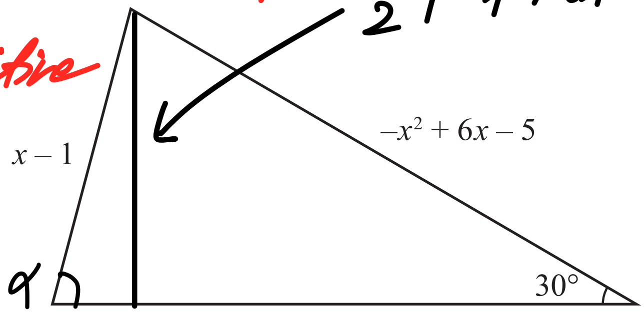
$$(x^2 - \sin^2\theta)(x^2 - \sin^2\beta) = 0.$$

$$-(\sin^2\theta + \sin^2\beta) = (\cos^2\theta - 1) + (\cos^2\beta - 1) = \frac{6}{7} - 2 = -\frac{8}{7}.$$

$$\sin^2\theta\sin^2\beta = (1 - \cos^2\theta)(1 - \cos^2\beta) = 1 - \frac{6}{7} + \frac{1}{7} = \frac{2}{7}$$

★ Basic conditions for triangle:
*longest side is shorter than sum of shorter sides.
* all sides are positive

17



Find the complete set of values of x for which there are two non-congruent triangles with the side lengths and angle as shown in the diagram.

A $1 < x < 3$

B $1 < x < 4$

C $1 < x < 5$

D $3 < x < 4$

E $3 < x < 5$

F $4 < x < 5$

9. triangle conditions.

$x-1 > \frac{1}{2}(-x^2+6x-5)$

$2x-2 > -x^2+6x-5$

$x^2-6x+5+2x-2 > 0$

$x^2-4x+3 > 0$

$x > 3$ or $x < 1$ … ①

Don't eliminate what's obvious.
Magnitude comparison must be explicit.

As each sides are positive: $x-1 > 0$, $-x^2+6x-5 > 0$

$(x-1)(x-5) < 0$

$1 < x < 5$ … ②

* $x-1 < -x^2+6x-5$

$(x-1) < (x-1)(5-x)$

$(x-1)(1-5+x) < 0$

$(x-1)(x-4) > 0$

$\therefore x < 4$ or $x > 1$ … ③

① ∩ ② ∩ ③ → $3 < x < 4$

learn how to sketch the polynomials (sketch end behaviours & pay attention to intercepts).

18 It is given that

$$f(x) = x^2(x-1)^2(x-2)$$

$$g(x) = -p(x-q)^2(x-r)^2$$

) p is positive & two positive roots.

where p, q and r are positive and $q < r$

Find the set of values of q and r that guarantees the greatest number of distinct real solutions of the equation $f(x) = g(x)$ for all p.

A $q < 1$ and $r < 1$

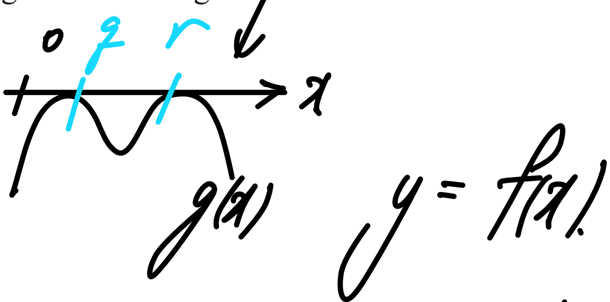
B $q < 1$ and $1 < r < 2$

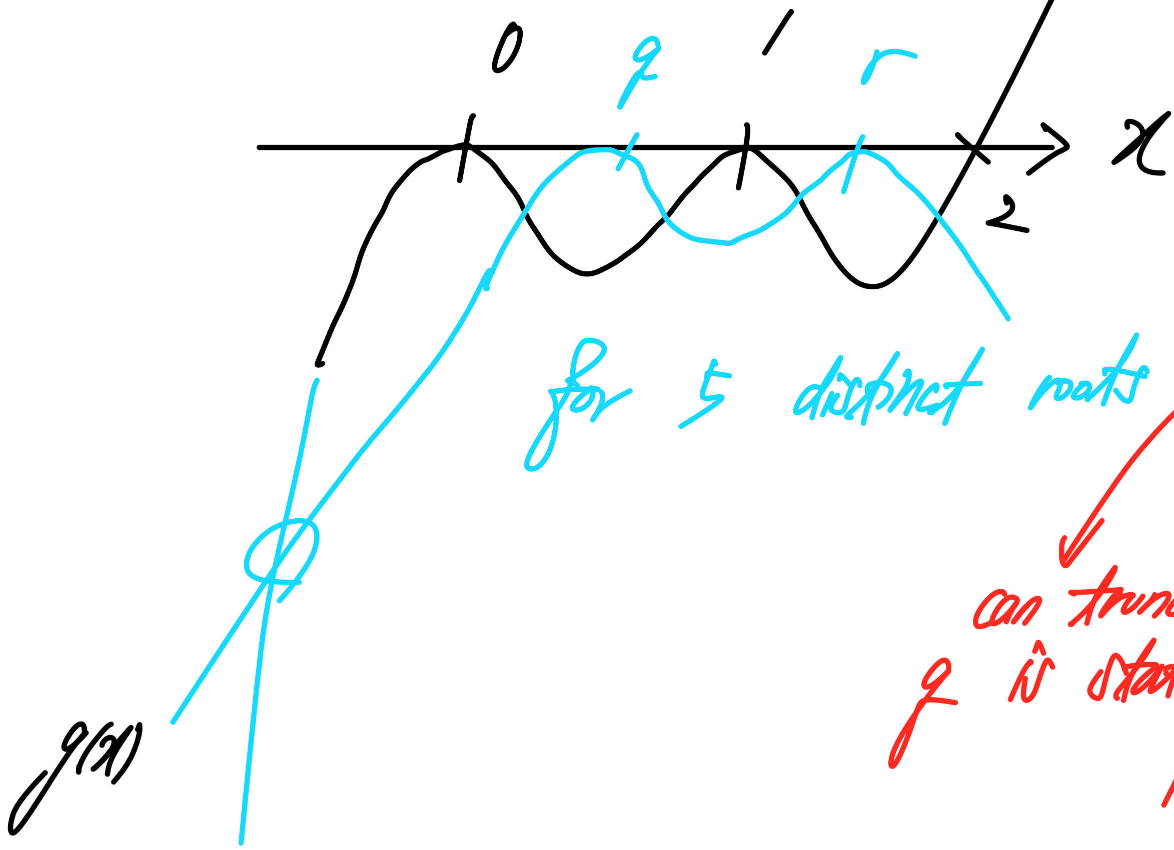
C $q < 1$ and $r > 2$

D $1 < q < 2$ and $1 < r < 2$

E $1 < q < 2$ and $r > 2$

F $q > 2$ and $r > 2$





for 5 distinct roots { $0 < q < 1$, $1 < r < 2$.

can truncate as q is stated as positive.

★ Excluding is super common in calculating probability.
(complementary set (truth set of negated proposition))

19 Circle C_1 is defined as $x^2 + y^2 = 25$

A second circle C_2 has radius 4 and centre (a, b) where

$$-2 \leq a \leq 2 \quad \text{and} \quad -3 \leq b \leq 3$$

If the centre of C_2 is equally likely to be located anywhere within the given range, what is the probability that C_2 intersects C_1?

A $\frac{1}{25}$

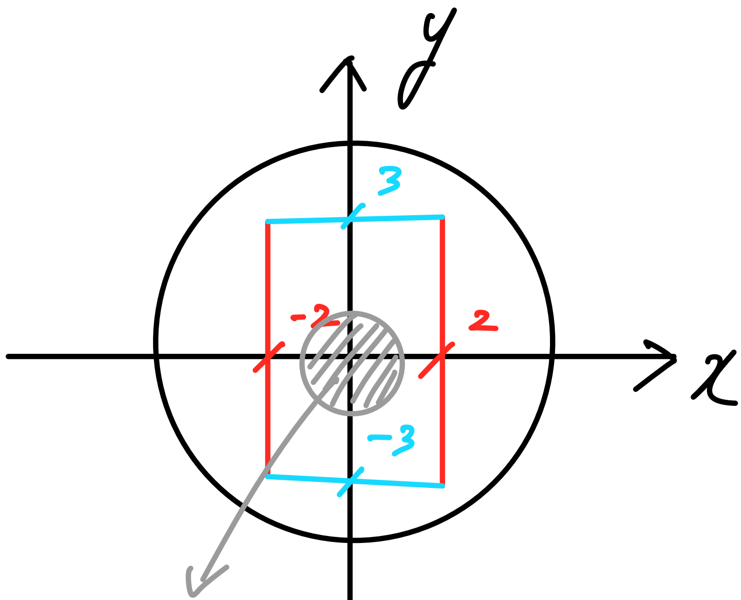
B $\frac{9}{25}$

C $\frac{16}{25}$

D $\frac{6-\pi}{6}$

E $\frac{16-\pi}{24}$

F $\frac{24-\pi}{24}$



if the centre of C_2 is within the grey-shaded area, what ends up happening is that they have no intersections.

(When C_2's centre is at the circumference of this unit circle $x^2+y^2=1$, the C_2 is inscribed to C_1.

$\rightarrow$ Total $= (2-(-2)) \times (3-(-3)) = 4 \times 6 = 24$

Excluding $1^2\pi = \pi$.

$1 - \frac{\pi}{24} \Rightarrow \frac{24-\pi}{24}$

20 n is the number of points of intersection of the graphs

B.

$$y = |x^2 - a^2| \text{ and } y = a^2|x - 1|$$

where a is a real number.

I) $a \neq 0$.

What is the smallest value of n that is **not** possible?

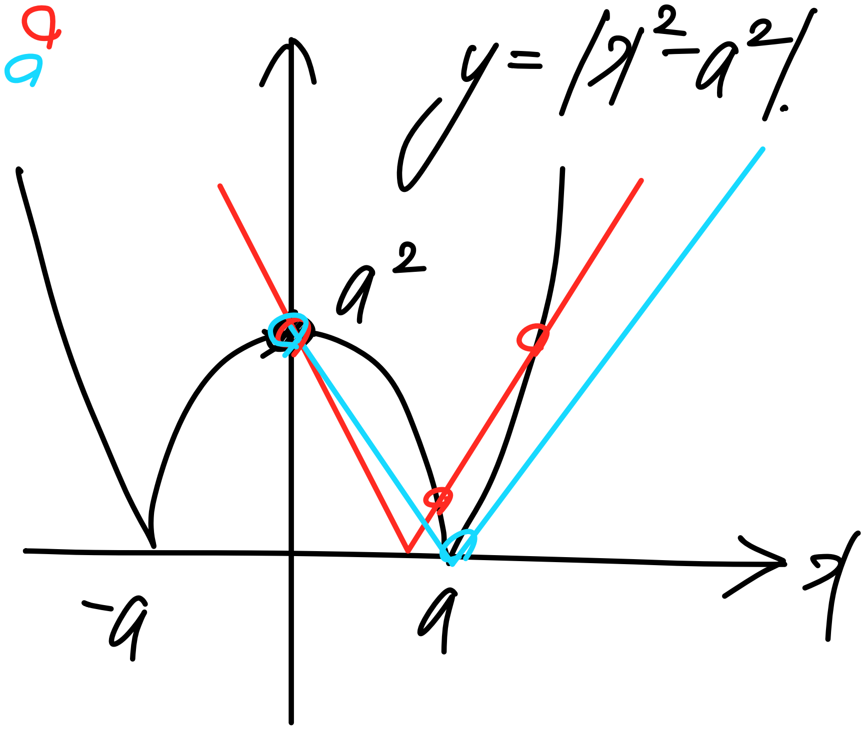
A $n = 1$ O

B $n = 2$ X

C $n = 3$ O

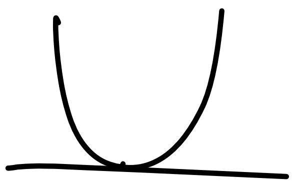
D $n = 4$ O

E $n = 5$ X



ii) $a = 0$.

$y = x^2$, $y = 0$



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