Cambridge Assessment Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION

PAPER 2 **D513/02**

2022 **75 minutes**

6213658757

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one. You also require a soft pencil and an eraser.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

This paper is the second of two papers.

This paper contains 20 questions. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

You can use the question paper for rough working or notes, but **no extra paper** is allowed.

You **must** complete the answer sheet within the time limit.

Calculators and dictionaries are NOT permitted.

There is no formulae booklet for this test.

TMUA9. Analysis.

Please wait to be told you may begin before turning this page.

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※ Only terms in $f'(x)$ that could be negative can influence the sign of $f'(x)$

1 Determine the number of stationary points on the curve with equation

$$y = 3x^4 + 4x^3 + 6x^2 - 5$$

A 0

B 1

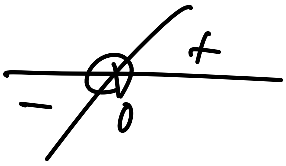
C 2

D 3

E 4

$y' = 12x^3 + 12x^2 + 12x.$

$12x(\underbrace{x^2+x+1}_{\oplus})$



※ only sign of x determines the sign of $f'(x)$. here $\because 12(x^2+x+1) > 0$

$\ast \sum_{k=0}^{n} {}_nC_k = {}_nC_0 + {}_nC_1 + \cdots + {}_nC_{n-1} + {}_nC_n = 2^n$

9. Combinatorics summation formulae (exhaustive)

2 Find the coefficient of the x^5 term in the expansion of

$$(1+x)^5 \times \sum_{i=0}^{5} x^i$$

A 1

B 5

C 16

D 25

E 32

$\underbrace{(1+x)^5}_{A}(\underbrace{x^0 + x^1 + \cdots}_{B} + x^5)$.

$$\left.\begin{aligned}(A,B) = (x^0, x^5) &\rightarrow {}_5C_0 \times 1\\ (x^1, x^4) &\rightarrow {}_5C_1 \times 1\\ &\vdots\\ (x^5, x^0) &\rightarrow {}_5C_5 \times 1\end{aligned}\right\}$$

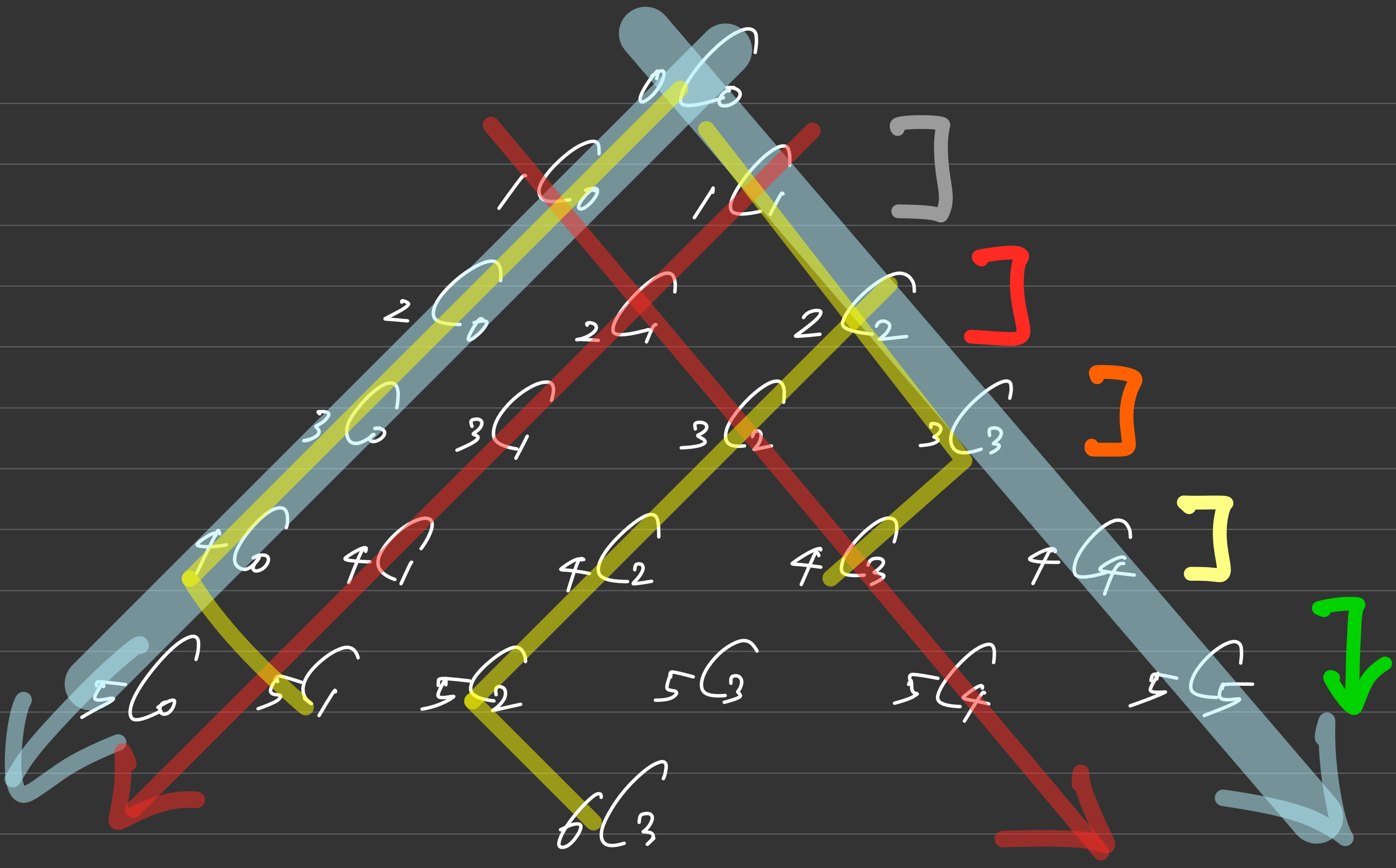
$\therefore \sum_{k=0}^{5} {}_5C_k = 2^5$

9. Combination Summation Formulae & Pascal's triangle.

The following is a Pascal's triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Let's examine the utility of them.



☆ The sides of the pascal triangle are always 1.
$({}^{n}C_0 = {}^{n}C_n = 1)$

☆ The second-inside sides of the pascal triangle (that Red thingy) is gonna be n itself. $(\because {}^{n}C_1 = {}^{n}C_{n-1} = n)$

☆ Hockey stick identity.

$$\sum_{k=r}^{n} {}^{k}C_r = {}^{n+1}C_{r+1}$$

Praxis 1) Coefficients of Binomial expansion.

$$(1+x)^1 = 1+x = {}_1C_1(1)^1(x)^0\ {}_1C_0(1)^0(x)^1$$

$$(1+x)^2 = 1+2x+x^2 = {}_2C_0(1)^2(x)^0 + {}_2C_1(1)^1(x)^1 + {}_2C_2(1)^0x^2$$

$$(1+x)^3 = 1+3x+3x^2+x^3 = {}_3C_0(1)^3(x)^0 + {}_3C_1(1)^2(x)^1 + {}_3C_2(1)^1(x)^2 + {}_3C_3(1)^0(x)^3$$

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

$$= {}_4C_0(1)^4(x)^0 + {}_4C_1(1)^3(x)^1 + {}_4C_2(1)^2(x)^2 + {}_4C_3(1)^1(x)^3 + {}_4C_4(1)^0(x)^4$$

$$(1+x)^n = 1 + \cdots + x^n = {}_nC_0(1)^n(x)^0 + {}_nC_1(1)^{n-1}(x)^1 + {}_nC_2(1)^{n-2}(x)^2 + \cdots + {}_nC_{n-1}(1)^1(x)^{n-1} + {}_nC_n(1)^0(x)^n.$$

(2) Praxis into summation of combinatorics.

let $f(x) = (1+x)^n$.

$* f(1)$: ${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{n-1} + {}_nC_n = (1+1)^n = 2^n$

* Caution!! it starts from ${}_nC_0$, not ${}_nC_1$. So $\sum_{k=1}^{n} {}_nC_k = 2^n - 1$ //

$\therefore \sum_{k=0}^{n} {}_nC_k = 2^n$: e.g). ${}_4C_0 + {}_4C_1 + {}_4C_2 + {}_4C_3 + {}_4C_4 = 2^4 = 16$!!

depends on if n is even/odd.

$* f(-1)$: ${}_nC_0 - {}_nC_1 + {}_nC_2 + \cdots + {}_nC_k(-1)^{n-k}(x)^k + \cdots \pm {}_nC_n x^n = (1-1)^n = 0$ //

e.g). ${}_5C_0 - {}_5C_1 + {}_5C_2 - {}_5C_3 + {}_5C_4 - {}_5C_5 = 0.$

either way.

$* \frac{1}{2}(f(1)+f(-1)) = {}_nC_0 + {}_nC_2 + {}_nC_4 + \cdots \left(+ {}_nC_{n-2} + {}_nC_n \right)$ if even $\left(\text{or} + {}_nC_{n-3} + {}_nC_{n-1}\right)$ if odd. $= \frac{1}{2}(2^n + 0) = 2^{n-1}$

e.g). ${}_7C_0 + {}_7C_2 + {}_7C_4 + {}_7C_6 = 2^{7-1}$

$* \frac{1}{2}(f(1)-f(-1)) = {}_nC_1 + {}_nC_3 + {}_nC_5 + \cdots + \left({}_nC_{n-2} + {}_nC_n\right)$ if odd. $\left(\text{or } {}_nC_{n-3} + {}_nC\right)$ if even. $= \frac{1}{2}(2^n - 0) = 2^{n-1}.$

e.g) ${}_9C_1 + {}_9C_3 + {}_9C_5 + {}_9C_7 + {}_9C_9 = 2^{9-1}$

9. 11^n.

- let's continue setting $f(x)$ as

for $f(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

$f(10) = (1+10)^n = 11^n$.

$$\sum_{k=0}^{n} \binom{n}{k} 10^k$$

$$= {}_nC_0 \cdot 10^0 + {}_nC_1 \times 10^1 + {}_nC_2 \times 10^2 + \cdots + {}_nC_k\, 10^k + \cdots + {}_nC_{n-1}\, 10^{n-1} + {}_nC_n\, 10^n.$$

at $n=1$: $(10+1)^1 = {}_{10}C_0\, 10^0 + {}_{10}C_1\, 10^1$
$= 1\times 1 + 10\times$

$1 \rightarrow 1 = 11^0$

$1 \quad 1 \rightarrow 11 = 11^1$

$1 \quad 2 \quad 1 \rightarrow 121 = 11^2$

$1 \quad 3 \quad 3 \quad 1 \rightarrow 1331 = 11^3$

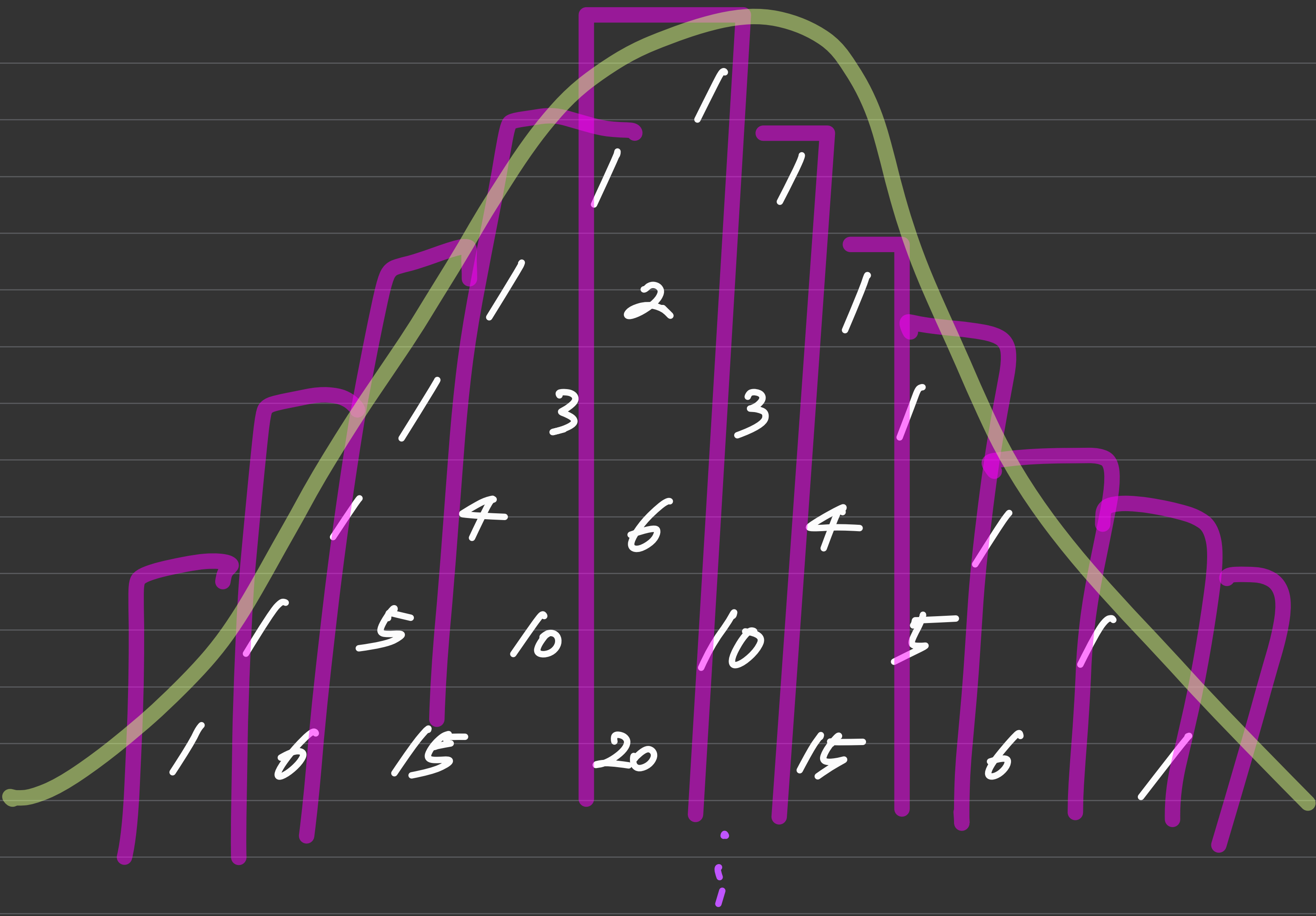
$1 \quad 4 \quad 6 \quad 4 \quad 1 \rightarrow 14641 \rightarrow 11^4$

$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$

here is where it kinda gets tricky, cause the combinatorics itself aren't singular digits.

→ Hence the combinatorics' value overflow into adjacent values.



$nC_0 \quad nC_1 \quad nC_2 \quad \cdots\cdots \quad nC_{n-1} \quad nC_n$

$\Downarrow$

If you visualise this via height, at n being large (so $n \to \infty$), it becomes a bell curve. So a normal distribution. (AKA Gaussian Distribution)

☆ Counter example of if ☆ then △ : If ☆ then not △ : Assume true condition, then not statement. (negated statement)

3 Consider the following statement about the positive integer n

$$\underbrace{\textbf{if } n \text{ is prime,}}_{☆} \textbf{ then } \underbrace{n^2+2}_{△} \text{ is } \textbf{not} \text{ prime}$$

Which of the following is a **counterexample** to this statement?

I $n = 2$ ☆ → △ (X)

II $n = 3$ ☆ → ~△ (O)

III $n = 4$ ~☆ → ~△ (X)

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Circle → Complete squares.

4 The point P has coordinates (p, q), and the equation of a circle is

$$x^2 + 2fx + y^2 + 2gy + h = 0$$

where f, g, h, p and q are all real constants.

Let L be the distance between the centre of the circle and the point P.

Which one of the following is **sufficient** on its own to be able to calculate L?

A the values of f, g and h

B the values of f, g, p and q

C the values of f, h, p and q

D the values of g, h, p and q

E none of the options **A-D** is sufficient on its own

$$(x+f)^2 - f^2 + (y+g)^2 - g^2 + h = 0.$$

$$(x+f)^2 + (y+g)^2 = f^2 + g^2 - h$$

L = Distance between (p, q) and $(-f, -g)$.

→ We need p, q, f, g.

* The problem with just trial-and-error is that you not finding the specific valid scenario is not a mathematically sufficient reason to find the solution

9. Propositions & Truth Sets.

5 A straight line L passes through $(1, 2)$.

Let P be the statement

if the y-intercept of L is negative, **then** the x-intercept of L is positive.

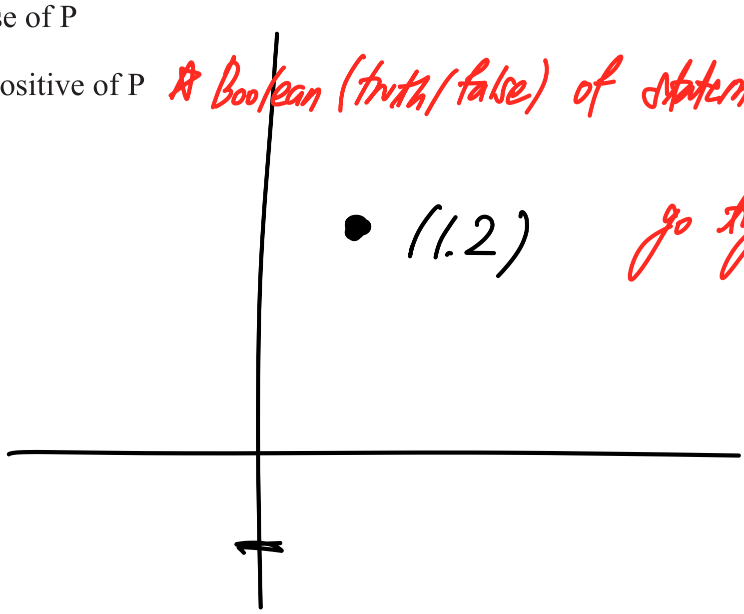
Which of the following statements **must** be true?

I P

II the converse of P

III the contrapositive of P

* Boolean (truth/false) of statement & its contrapositive go together.



A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I & III: $y = m(x-1) + 2$.

at $x = 0$: $y = -m+2$.

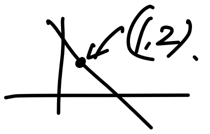
Hence, y intercept: $(0, 2-m)$ (Assuming the condition is true)

If $2 - m < 0$, $(2 < m)$

x intercept: $m(x-1) = -2$.

$x = 1 - \frac{2}{m}$

II: Counterexample. $q \to p$



As $m > 2$, $m > 0$ so $1 > \frac{2}{m}$

$-1 < -\frac{2}{m}$

$0 < 1 - \frac{2}{m}$. thus

$0 <$ y-intercept.

9. logic.

A: $p \to q$.

Converse of A = $q \to p$

inverse of A = $\sim p \to \sim q$.

Contrapositive of A = $\sim q \to \sim p$.

Truth statements

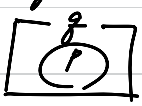
☆ The inverse of converse is necessary and sufficient to the converse of the inverse.

✓ Inverse of converse of A: $(p \to q) \xrightarrow{\text{Converse}} (q \to p) \xrightarrow{\text{Inverse}} (\sim q \to \sim p)$.

✓ Converse of inverse of A: $(p \to q) \xrightarrow{\text{Inverse}} (\sim p \to \sim q) \xrightarrow{\text{Converse}} (\sim q \to \sim p)$

They're all interchangable thus exactly the same thing!

If A: $p \to q$, then everything below is true.! (condition = p, statement = q)

- Inverse of converse of A
- Converse of inverse of A
- Contrapositive.
- If negated statement is true, then negated condition is true.
- If condition is true, then the statement is true
- If condition, then statement.
- Statement (is true), if condition (is true)
- $p \to q$
- $\sim q \to \sim p$
- 

9. Propositions and Truth Sets.

* **Propositions** = A sentence that is strictly either true or false. (usually written in lowercase. like p.)

→ in layman terms; it's a claim. So like booleans in Python, they are something of which an assessment of it being true/false can be judged.

* **Truth set** = Collection of all elements that make that sentence true. (Usually written in uppercase. like P. A.)

→ It's an exhaustive list/dump. Pooled elements that make that true.

e.g1) p: y is a planet in the Solar System.

P = {Mars, Earth, Jupiter... }.

e.g2) q: x is an even number.

Q: { x = 2, 4, 6, ... }.

* Assess both ($p \to q$, $q \to p$.) attitude. Be on the look for counterexamples.

6 A list consists of n integers.

THE ONLY EXCUSE for not finding a counterexample must be mathematical validity.

Consider the following statements:

P: n is odd.

Q: The median of the list is one of the numbers in the list.

Which one of the following is true?

A P is **necessary and sufficient** for Q.

B P is **necessary** but **not sufficient** for Q.

C P is **sufficient** but **not necessary** for Q.

D P is **not necessary** and **not sufficient** for Q.

1) $P \to Q$ (P is sufficient).

$a_1 \quad a_2 \quad \dots \quad a_{k-1} \quad a_k \quad a_{k+1} \quad \dots \quad a_{2k-1}$

a_k → median

$n = 2k-1$ (odd)

2) $Q \to P$. (P is necessary)

Counterexample: let $n = 4$, $\neq$ odd.

$a_1 \quad a_2 \quad a_3 \quad a_4$

$-1 \quad 2 \quad 2 \quad 869$

→ median 2

7 Consider the following claim:

The difference between two consecutive positive cube numbers is always prime.

Here is an attempted proof of this claim:

I $(x+1)^3 = x^3 + 3x^2 + 3x + 1$

II Taking x to be a positive integer, the difference between two consecutive cube numbers can be expressed as $(x+1)^3 - x^3 = 3x^2 + 3x + 1$

III It is impossible to factorise $3x^2 + 3x + 1$ into two linear factors with integer coefficients because its discriminant is negative.

~~IV~~ Therefore for every positive integer value of x the integer $3x^2 + 3x + 1$ cannot be factorised. → here factorisation means. prime factorisation.

V Hence, the difference between two consecutive cube numbers will always be prime.

Which of the following best describes this proof?

A The proof is completely correct, and the claim is true.

B The proof is completely correct, but there are counterexamples to the claim.

C The proof is wrong, and the first error occurs on line I.

D The proof is wrong, and the first error occurs on line II.

E The proof is wrong, and the first error occurs on line III.

F The proof is wrong, and the first error occurs on line IV.

G The proof is wrong, and the first error occurs on line V.

Why? Why does the term factorisation here indicate prime factorisation, not factors in polynomials' expansions?

Because they said "integer" & cannot be factorised. If it was instead stated as function in terms of x that would NOT mean prime factorisation.

9. Discernment between technical terms.

"factorisation" can mean two things.

① Prime factorisation. ═══ (subject = number)

e.g) $4 = 2^2$

$1001 = 7 \times 11 \times 13$.

② Factor Multiplication. ═══ (subject: mathematical expression; Commonly polynomials)

$x^2 - 5x + 6 = (x-2)(x-3)$.

$x^4 - 6x^3 + 5x^2 = x^2(x-1)(x-5)$

→ How do I discern which one it is indicating?

9. → Identify the "verbal expression" as the subject.

Here, it said, "the integer" / $3x^2+3x+1$ / can't be factorised. You might ask, "yeah but what about $3x^2+3x+1$? that's a polynomial!"

That's why I said VERBAL 😉

★ You might get this right by accident, fix it

8 A selection, S, of n terms is taken from the arithmetic sequence 1, 4, 7, 10, … , 70.

Consider the following statement:

$(*)$ There are two distinct terms in S whose sum is 74.

What is the smallest value of n for which $(*)$ is **necessarily** true?

A 12

B 13

C 14

D 21

E 22

F 23

i) at $n=1$. (start with extremes).

$\{1\}$.

at $n=2$.

$\{1,4\} = \{3(1)-2, 3(2)-2\}$.

⋮

at $n=13$.

$\{3(1)-2, 3(2)-2, \dots 3(13)-2\}$

$3(12)-2 = 34$.

$= 37$.

at $n=13$ is the tipping point. $\max(a_m+a_n)$ at the two sets are $34+37=71$. (still short by 3 for 74)

at $n=14$

$\{3(1)-2, 3(2)-2, \cdot\cdot\, 3(13)-2, 3(14)-2\}$.

34, 37, 40.

max: $(34+40)$.

ii)

*** You also need to go the other way & compare. The previous was only checking ascending orders: so previous method was adding elements from ascending order. then finding maximum value

$\rightarrow u_n = 3n-2,\ u_n = 70,\ n=24$.

$u_{24} = 70,\ u_{23} = 67,\ u_{22} = 64 \dots$ needs to go to u_{14}, u_{13}, u_{12}

$24-12+1 = 13. \longrightarrow$ comparing (i) and (ii) for necessary, $\max(13, 14) = 14$.

* for inequalities, consider boundary conditions independently.
* Start inequalities' interpretations with boundaries first, not last.

9 Consider the following statement:

$$(*) \textbf{ For all } \text{real numbers } x, \textbf{ if } x < k \textbf{ then } x^2 < k$$

What is the complete set of values of k for which $(*)$ is true?

A no real numbers

B $k > 0$

C $k < 1$

D $k \leq 1$

E $0 < k < 1$

F $0 < k \leq 1$

G all real numbers

sol idea) USE MCQ for your advantage.

① at $x = 0$: $x < 0$, $\to x^2 < 0$ (X) try $x = -100$ trillion.

② at $x = 1$. $x < 1$, $\to x^2 < 1$ (X). likewise.

* then articulate the general form of these counter-examples.

* "There exists" are similar to counterexamples in that you only need to win once.
↳ only need to prove ∠ exists. ↳ only need to prove ∠ does not exist.

10 Which of the following statements is/are true?

I **For all** real numbers x and **for all** positive integers n, $x < n$

II **For all** real numbers x, **there exists** a positive integer n such that $x < n$

III **There exists** a real number x such that **for all** positive integers n, $x < n$

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

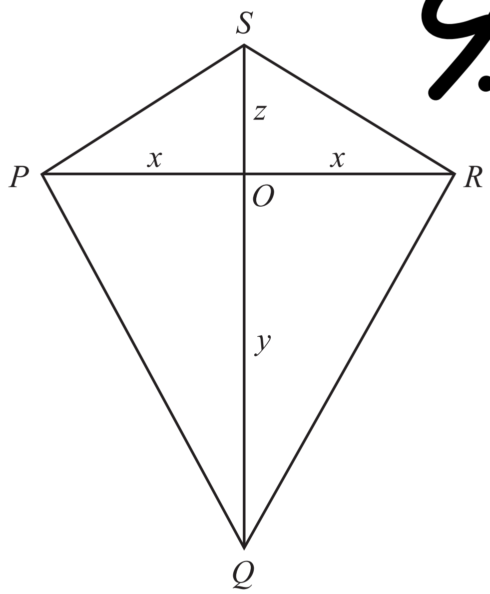
'I' states that every single positive integer must be greater than every real number. Hell naw.

III. Obviously, $x=0$.

Inter triangle formula in TMUA 9. Noir Premiere.

11

9. Triangle Ratio Recap.



The diagram shows a kite *PQRS* whose diagonals meet at *O*.

$OP = x$
$OQ = y$
$OR = x$
$OS = z$

Which of the following is **necessary and sufficient** for angle *SPQ* to be a right angle?

A $x = y = z$

B $2x = y + z$

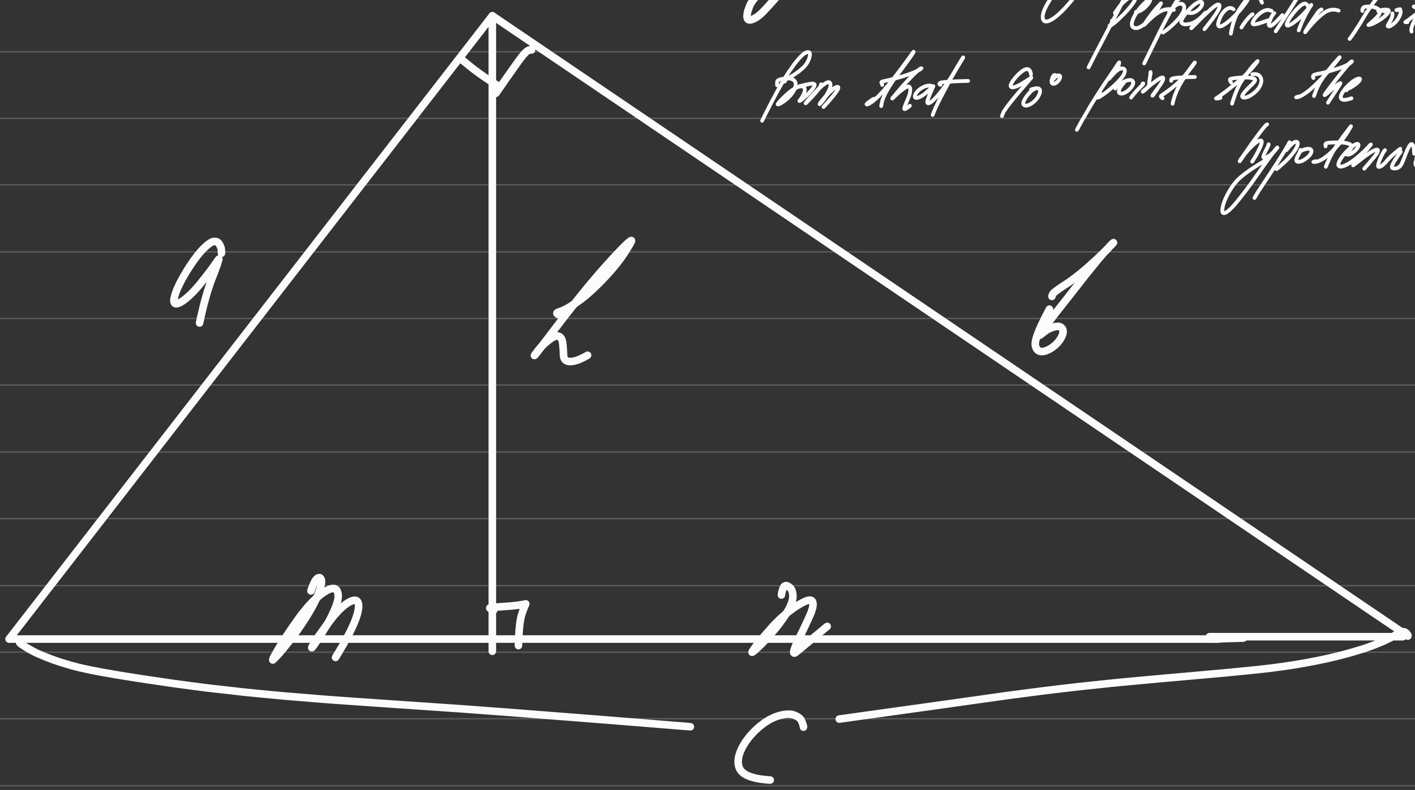
C $x^2 = yz$

D $y = z$

E $y^2 = x^2 + z^2$

9. Recap. ↗: Geometry. Triangles.

Condition: Right Triangle. (The big one) and a perpendicular foot from that 90° point to the hypotenuse.



- $ab = ch.$
- $a^2 = mc$
- $b^2 = nc.$
- $h^2 = mn$

(cf) $mb^2 + na^2 = c(h^2 + mn)$. from Stewart theorem. Check TMVA9.

Noir Premiere.

9. Definition of increasing function (graphs) & the intuition of integrals.

12 Place the following integrals in order of size, starting with the smallest.

$$P = \int_0^1 2^{\sqrt{x}}\,dx$$

$$Q = \int_0^1 2^{x}\,dx$$

$$\int_0^1 2^{\frac{1}{2}x}\,dx = R = \int_0^1 (\sqrt{2})^{x}\,dx$$

A $P < Q < R$

B $P < R < Q$

C $Q < P < R$

D $Q < R < P$

E $R < P < Q$

F $R < Q < P$

As 2^x is increasing function AND! one-to-one function,

if $x_1 < x_2 < x_3$, then $f(x_1) < f(x_2) < f(x_3)$.

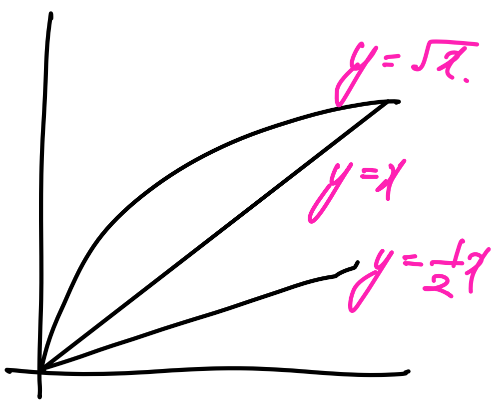
Converse is also true.

$\therefore$ If $* < \Delta$, then $2^* < 2^\Delta$.

⊕ $2^x > 0$. Hence, if $2^* < 2^\Delta$, then $\int 2^* < \int 2^\Delta$.

So to compare magnitude of P, Q, R

$\rightarrow$ compare magnitude of $\sqrt{x}$, x, $\frac{1}{2}x$. AT $x \in [0,1]$.



Hence, $\sqrt{x} > x > \frac{1}{2}x$

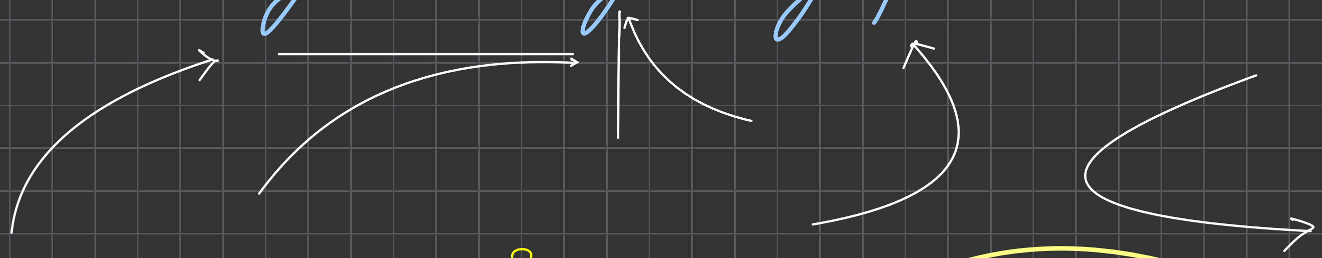
$\therefore P > Q > R$

$R < Q < P$

9. Mathematical Definition of an Increasing Decreasing function.

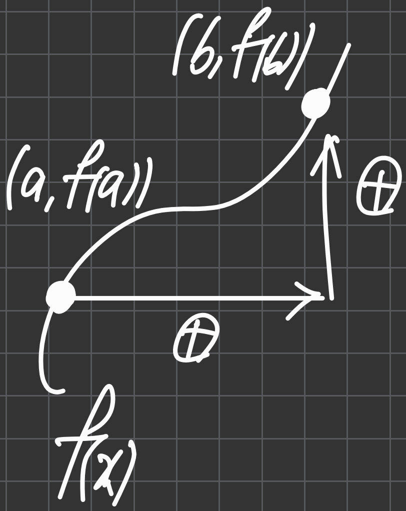
* you can't just vaguely say f(x) is going up so it's a

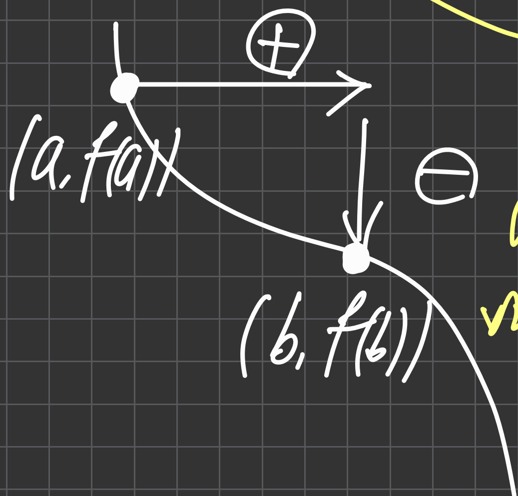
A key attribute of being an elite mathematician is to be able to succinctly articulate what you intend when you use any expression to describe something.



What is increasing or decreasing?

- $f(x)$ is increasing $\iff$ $\left\{ \text{If } x=a,b \in \text{Domain of } f(x), \text{ if } a<b, \text{ then } f(a)<f(b). \right\}$
- $f(x)$ is decreasing $\iff$ $\left\{ \text{If } x=a,b \in \text{Domain of } f(x), \text{ if } a<b, \text{ then } f(a)>f(b). \right\}$





you can constrain a Domain that is a subset of a larger domain and say it increases.

cf) try $h(x) = \left(\sqrt[3]{e^{\pi}}\right)^{\log_{19}(x^2+4x+9.7)}$

$h(x) = f_1(f_2(f_3(x)))$, s.t $\begin{cases} f_1(x) = \left(\underline{\sqrt[3]{e^{\pi}}}\right)^{x} = a^x \text{ form.} \\ \quad (a>1) \\ f_2(x) = \log_{19}(x+5.7) = \log_a x \text{ form.} \\ f_3(x) = (x+2)^2 \geq 0. \\ \quad = \text{Quadratic.} \end{cases}$

look at what's obvious.

Comparing. $\int_0^1 h(x)\,dx$ & $\int_0^1 f_1(f_2(4x))\,dx$.

$f_1(x) = a^x$ s.t $a>1$ $\left(\because \sqrt[3]{e^{\pi}} > 1\right)$, so ($f_1(x)$ is an increasing function with $f_1(x)>0$ &

$f_2(x) = \log_a x$ s.t $a>1$ and $x>0$, $f_2(x) \in (-\infty, \infty)$ one-to-one function.

- As $f_1(x)$ is increasing if $☆ < \triangle$ then $f_1(☆) < f_1(\triangle)$

As $f_1(x) > 0$, if $f_1(☆) < f_1(\triangle)$, then $\int_a^b f_1(☆)\,dx < \int_a^b f_1(\triangle)\,dx$.

s.t $a<b$, $a, b \in \mathbb{R}$

$\downarrow$

So Magnitude of $\int_0^1 f_1(f_2(f_3(x)))$ vs $\int_0^1 f_1(f_2(4x))dx$.

is same as magnitude of $f_3(x)$ & $4x$.

$\rightarrow$ As $(x+2)^2 - 4x$.

$= x^2 + 4 > 0$,

$\int_0^1 h(x)\,dx > \int_0^1 f_1(f_2(x))\,dx$

13 Consider the statement (∗) about a real number x:

(∗) **There exists** a real number y such that $x - xy + y$ is negative.

For how many real values of x is (∗) true?

A no values of x

B exactly one value of x

C exactly two values of x

D all except exactly two values of x

E all except exactly one value of x

F all values of x

$xy - x - y > 0.$

$xy - x - y + 1 > 1$

$(x-1)(y-1) > 1.$

$y \neq 1.$ Hence $\exists x$ s.t

$(x-1)(y-1) > 1.$

$\therefore \forall y \ (y \neq 1), \exists x$ s.t $(x-1)(y-1) > 1$

14 Consider the two inequalities:

$$|x+5| < |x+11|$$

$$|x+11| < |x+1|$$

$|x+a|$ all consist of translated version of $y = \pm x$.//

Which one of the following is correct?

A There is no real number for which both inequalities are true.

B There is exactly one real number for which both inequalities are true.

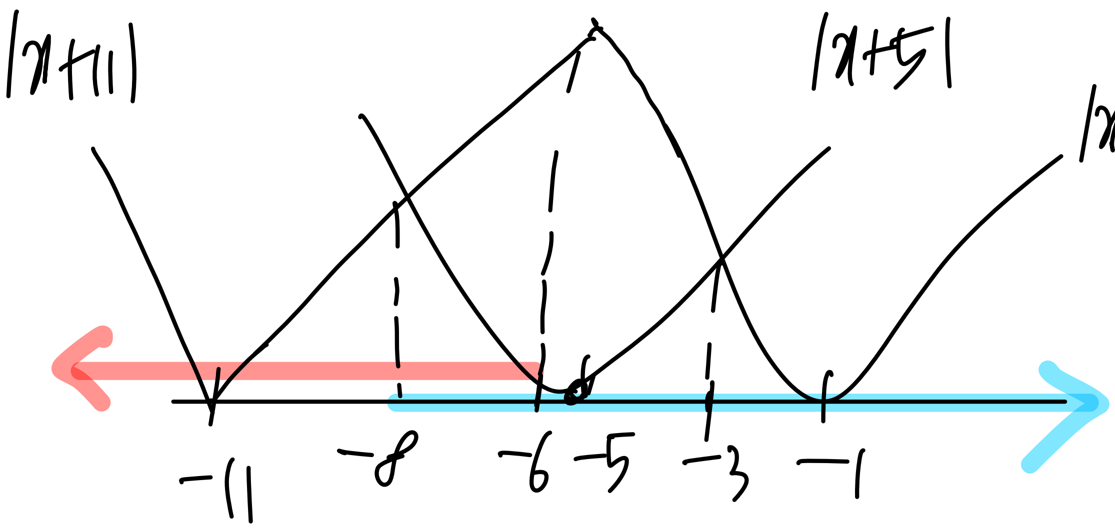
C The real numbers for which both inequalities are true form an interval of length 1.

D The real numbers for which both inequalities are true form an interval of length 2.

E The real numbers for which both inequalities are true form an interval of length 3.

F The real numbers for which both inequalities are true form an interval of length 4.

G The real numbers for which both inequalities are true form an interval of length 5.



15 The real numbers x, y and z are all greater than 1, and satisfy the equations

$$\log_x y = z \quad \text{and} \quad \log_y z = x$$

Which one of the following equations for $\log_z x$ **must** be true?

A $\log_z x = y$

B $\log_z x = \frac{1}{y}$

C $\log_z x = xy$

D $\log_z x = \frac{1}{xy}$

E $\log_z x = xz$

F $\log_z x = \frac{1}{xz}$

G $\log_z x = yz$

H $\log_z x = \frac{1}{yz}$

$x^z = y$

sub.

$y^x = z$

$(x^z)^x = z$

$\log_z$

$\log_z x^{zx} = 1$

$\log_z x = \frac{1}{xz}$

16 In this question, $a_1, \ldots, a_{100}$ and $b_1, \ldots, b_{100}$ and $c_1, \ldots, c_{100}$ are three sequences of integers such that

$$a_n \leq b_n + c_n$$

for each n.

$C_1 = -☆.$

$C_n = -☆.$ $C_2 = -99.$

$n + ☆ = 101$ ⋮

$☆$ $C_{100} = -100$

Which of the following statements **must** be true?

I (minimum of $a_1, \ldots, a_{100}$) $\leq$ (minimum of $b_1, \ldots, b_{100}$) $+$ (minimum of $c_1, \ldots, c_{100}$)

II (minimum of $a_1, \ldots, a_{100}$) $\geq$ (minimum of $b_1, \ldots, b_{100}$) $+$ (minimum of $c_1, \ldots, c_{100}$)

III (maximum of $a_1, \ldots, a_{100}$) $\leq$ (maximum of $b_1, \ldots, b_{100}$) $+$ (maximum of $c_1, \ldots, c_{100}$)

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

I. Counter: let $a_1 = a_2 = \cdots = a_{100} = -101$

let $b_n = -n$ $(n \in [1, 100],\ n \in \mathbb{N})$

$C_n = 101 - n$

$b_n + C_n = 101 - 2n.$

$n=1$: $b_1 + C_1 = 101 - 2 = 99 > -101$

$n=2$: $b_2 + C_2 = 101 - 4 = 97 > -101$

⋮

$n=100$: $b_{100} + C_{100} = 101 - 200 = -99 > -101$

min of $(a_1, a_2, \cdots a_{100}) = -101$

min of $(b_1, b_2, \cdots b_{100}) = -100.$

min of $(C_1, C_2, \cdots C_{100}) = -100.$

$-101 \geq 100 + 100.$ (X)

II: Counter: $a_n = -101$, $b_n = -n.$, $C_n = -n$

17 A student answered the following question:

> a and b are non-zero real numbers.
>
> Prove that the equation $x^3 + ax^2 + b = 0$ has three distinct real roots if $27b\left(b + \frac{4a^3}{27}\right) < 0$

Here is the student's solution:

I We differentiate $y = x^3 + ax^2 + b$ to get $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax = x(3x + 2a)$

Solving $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ shows that the stationary points are at $(0, b)$ and $\left(-\frac{2a}{3}, b + \frac{4a^3}{27}\right)$

II If $27b\left(b + \frac{4a^3}{27}\right) < 0$, then b and $b + \frac{4a^3}{27}$ must have opposite signs, and so one of the stationary points is above the x-axis and one is below.

* must be reversed.

→ the condition.

III If the cubic has three distinct real roots, then one of the stationary points is above the x-axis and one is below. → the conclusion.

IV Hence if $27b\left(b + \frac{4a^3}{27}\right) < 0$, then the equation has three distinct real roots.

Which one of the following options best describes the student's solution?

A It is a completely correct solution.

B The student has instead proved the converse of the statement in the question.

C The solution is wrong, because the student should have stated step II after step III.

D The solution is wrong, because the student should have shown the converse of the result in step II.

E The solution is wrong, because the student should have shown the converse of the result in step III.

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

9. Sketching composite functions.

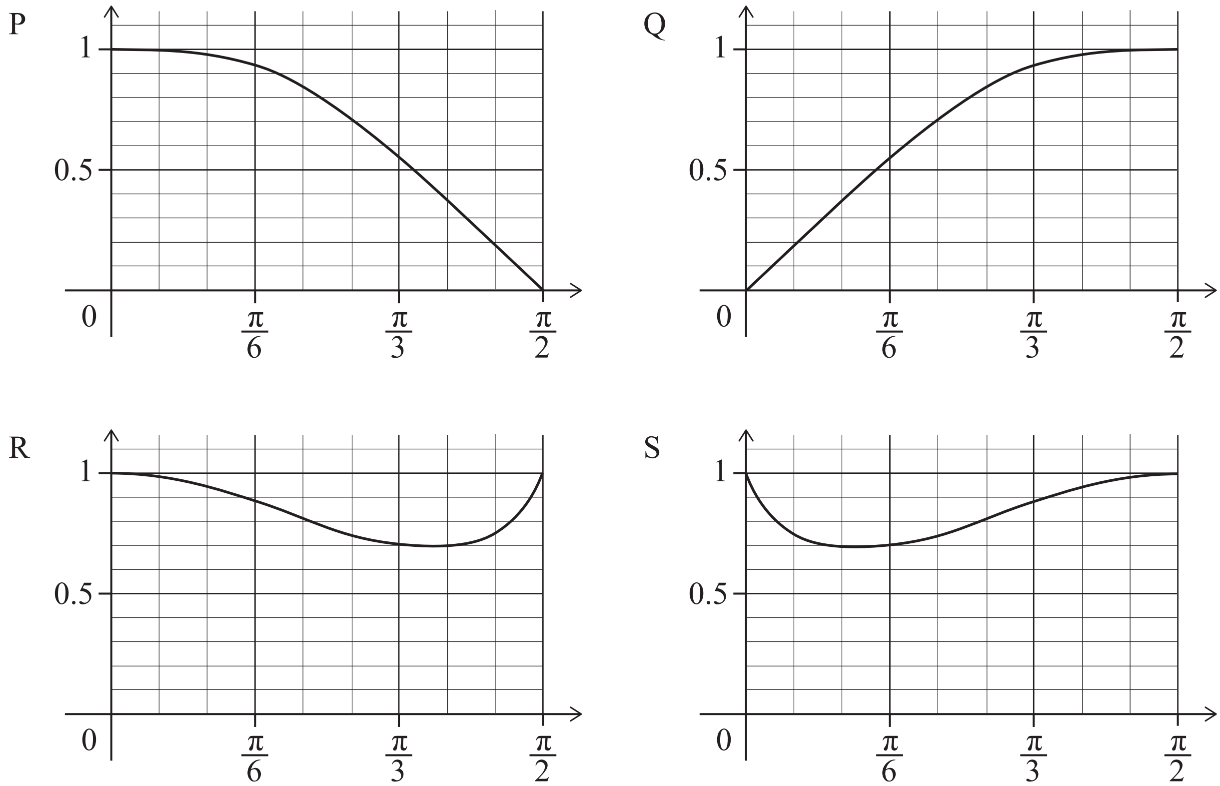
18 E

P, Q, R and S show the graphs of

$$y = (\cos x)^{\cos x},\quad y = (\sin x)^{\sin x},\quad y = (\cos x)^{\sin x} \text{ and } y = (\sin x)^{\cos x}$$

for $0 < x < \frac{\pi}{2}$ in some order.

sol idea) Compare. function value at $x = \frac{\pi}{6}$: $P > R > S > Q$.

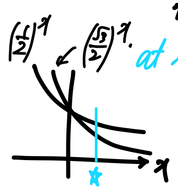


Which row in the following table correctly identifies the graphs?

	$y = (\cos x)^{\cos x}$	$y = (\sin x)^{\sin x}$	$y = (\cos x)^{\sin x}$	$y = (\sin x)^{\cos x}$
A	P	Q	R	S
B	P	Q	S	R
C	Q	P	R	S
D	Q	P	S	R
E	R	S	P	Q
F	R	S	Q	P
G	S	R	P	Q
H	S	R	Q	P

$\left(\frac{1}{2}\right)^{\frac{1}{2}}$ vs $\left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}$: $y = \left(\frac{1}{2}\right)^{x}$ decreasing. $\left(\frac{1}{2} < 1, \frac{\sqrt{3}}{2} < 1\right)$ so $\left(\frac{1}{2}\right)^{\frac{1}{2}} > \left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}$

$\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}}$ vs $\left(\frac{\sqrt{3}}{2}\right)^{\frac{\sqrt{3}}{2}}$: $y = \left(\frac{\sqrt{3}}{2}\right)^{x}$ $\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}} > \left(\frac{\sqrt{3}}{2}\right)^{\frac{\sqrt{3}}{2}}$.



at $x = k$ $(k > 0)$, $\left(\frac{1}{2}\right)^{k} > \left(\frac{\sqrt{3}}{2}\right)^{k}$.

$\therefore \left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}} > \left(\frac{1}{2}\right)^{\frac{1}{2}}$, $\left(\frac{\sqrt{3}}{2}\right)^{\frac{\sqrt{3}}{2}} > \left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}$, so $\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}} > \left(\frac{\sqrt{3}}{2}\right)^{\frac{\sqrt{3}}{2}} > \left(\frac{1}{2}\right)^{\frac{1}{2}} > \left(\frac{1}{2}\right)^{\frac{\sqrt{3}}{2}}$

★ 9. 「N-axis technique」

Skill for sketching composite functions.

The reality is that knowing the back-end mechanism, the 'why' behind sketching graphs are necessary. (MATLAB & Desmos isn't always there for you.)

(Check 「Noir Premiere」, it discusses the full guide on how to sketch composite functions from scratch.

9. $\sin☆ = \cos☆$ $\sin\triangle = \cos\triangle$

19 A polygon has n vertices, where $n \geq 3$. It has the following properties:

- Every vertex of the polygon lies on the circumference of a circle C.
- The centre of the circle C is inside the polygon.
- The radii from the centre of the circle C to the vertices of the polygon cut the polygon into n triangles of equal area.

For which values of n are these properties **sufficient** to deduce that the polygon is regular?

A no values of n

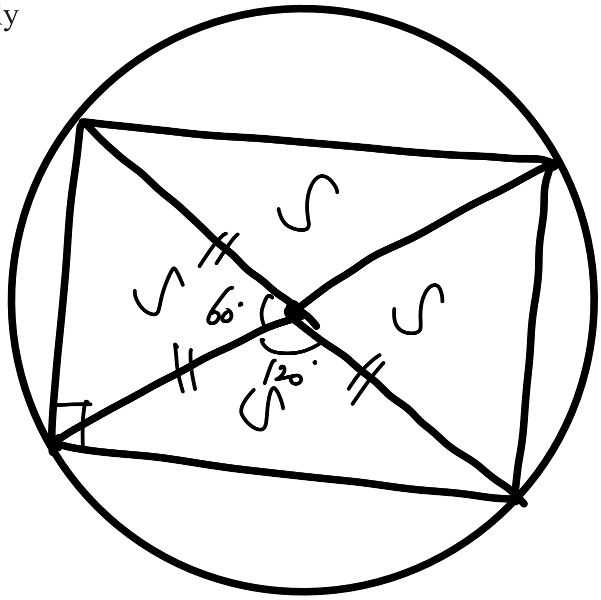
B $n = 3$ only

C $n = 3$ and $n = 4$ only

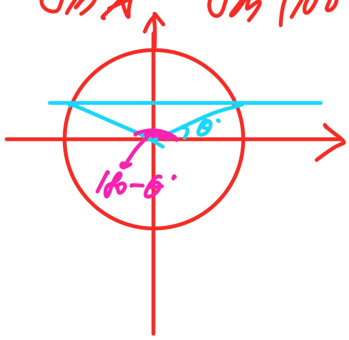
D $n = 3$ and $n \geq 5$ only

E all values of n

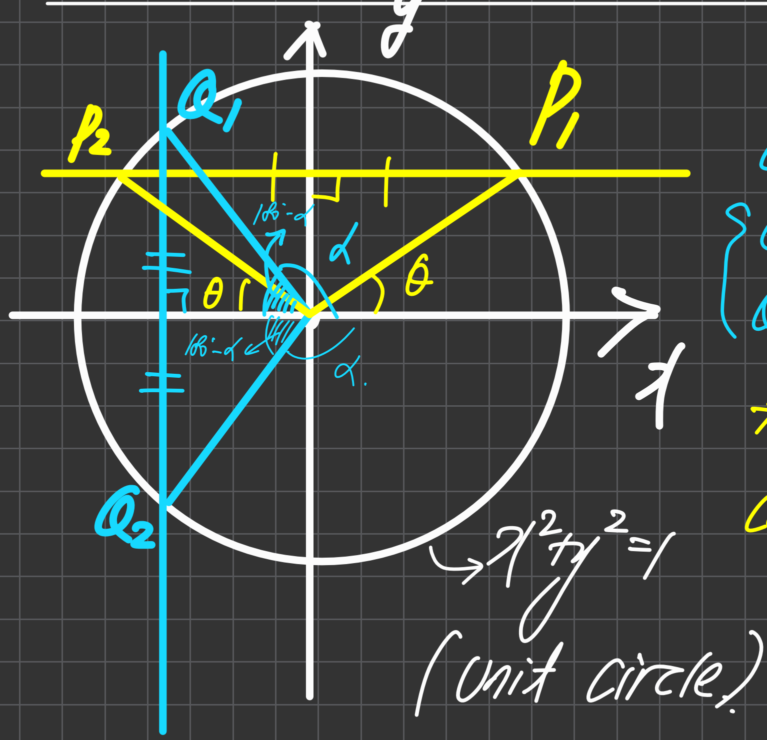
$n=4$:



generalization: why is $n=4$ out? $\Rightarrow$ $\sin☆ = \sin(180-☆)$



9. $\sin \star = \cos \star$, $\sin \Delta = \cos \Delta$.



* Q_1 & Q_2 has same x coordinates thus same cosine values

$\begin{cases} Q_1(\cos\alpha, \sin\alpha) \\ Q_2(\cos(360^\circ-\alpha), \sin(360^\circ-\alpha)) \end{cases}$

* P_1 & P_2 have same y coordinates thus same sine values.

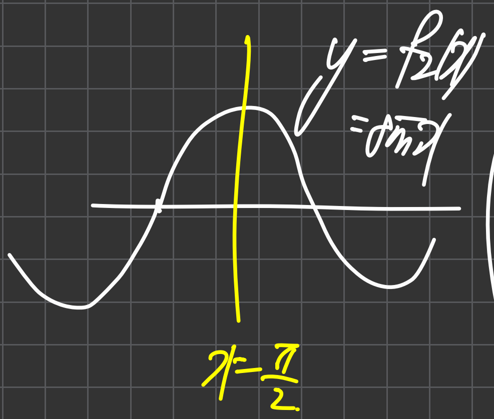
$\begin{cases} P_1(\cos\theta, \underline{\sin\theta}) \\ P_2(\cos(180^\circ-\theta), \underline{\sin(180^\circ-\theta)}) \end{cases}$

$\therefore \sin\theta = \sin(180^\circ-\theta)$

on the actual graph.

let $f_2(x) = \sin x$: $\underline{f_2(\theta) = f_2(\pi-\theta)}$

indicates that $f_2(x)$ is $x=\frac{\pi}{2}$ symmetric.



#Recursion #Composite functions.

29 The functions f_1 to f_5 are defined on the real numbers by

$$f_1(x) = \cos x$$
$$f_2(x) = \sin(\cos x)$$
$$f_3(x) = \cos(\sin(\cos x))$$
$$f_4(x) = \sin(\cos(\sin(\cos x)))$$
$$f_5(x) = \cos(\sin(\cos(\sin(\cos x))))$$

where all numbers are taken to be in radians.

These functions have maximum values m_1, m_2, m_3, m_4 and m_5, respectively.

Which one of the following statements is true?

~~A~~ m_1, m_2, m_3, m_4 and m_5 are all equal to 1

~~B~~ $0 < m_5 < m_4 < m_3 < m_2 < m_1 = 1$

~~C~~ $m_1 = m_3 = m_5 = 1$ and $0 < m_2 = m_4 < 1$

~~D~~ $m_1 = m_3 = m_5 = 1$ and $0 < m_4 < m_2 < 1$

E $m_1 = m_3 = 1$ and $0 < m_2 = m_4 < 1$ and $0 < m_5 < 1$

F $m_1 = m_3 = 1$ and $0 < m_4 < m_2 < 1$ and $0 < m_5 < 1$

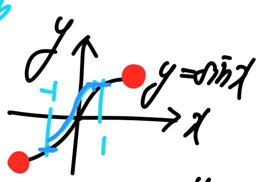
in exam. $m_2 = m_4$ so ok.

no need to actually find m_5

① $\max(f_1(x)) = 1. = m_1$

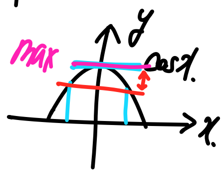
② $\max(f_2(x))$: $f_2(x) = \sin(f_1(x))$ s.t $-1 \le f_1(x) \le 1$

$\therefore \max(f_2(x))$: at $f_1(x) = 1$, $\sin(f_1(x)) = \sin(1) = m_2$



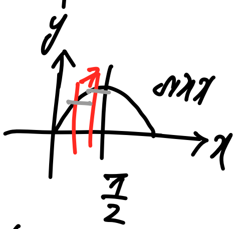
③ $\max(f_3(x))$: $f_3(x) = \cos(f_2(x))$ s.t $-\sin 1 \le f_2(x) \le \sin 1$
$(= \sin(-1))$

$\max(f_3(x)) = 1. = m_3$



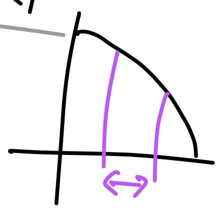
④ $\max(f_4(x))$: $f_4(x) = \sin(f_3(x))$ s.t $\cos(\sin(1)) \le f_3(x) \le 1 < \frac{\pi}{2}$

$\max(f_4(x)) = \sin(1) = m_4$



⑤ $\max(f_5(x))$: $f_5(x) = \cos(f_4(x))$ s.t $0 < \sin(\cos(\sin(1))) \le f_4(x) \le \sin 1 < 1$

$\max(f_5(x)) = f_5(\sin(\cos(\sin(1)))) < 1. = m_5$



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