TMUA9. Model Answer.

Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION **D513/11**

PAPER 1

Wednesday 31 October 2018 Time: 75 minutes

Additional materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt **all** 20 questions. Each question is worth one mark.

Any rough work should be done on this question paper. No extra paper is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

Calculators and dictionaries must **NOT** be used.

There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV1

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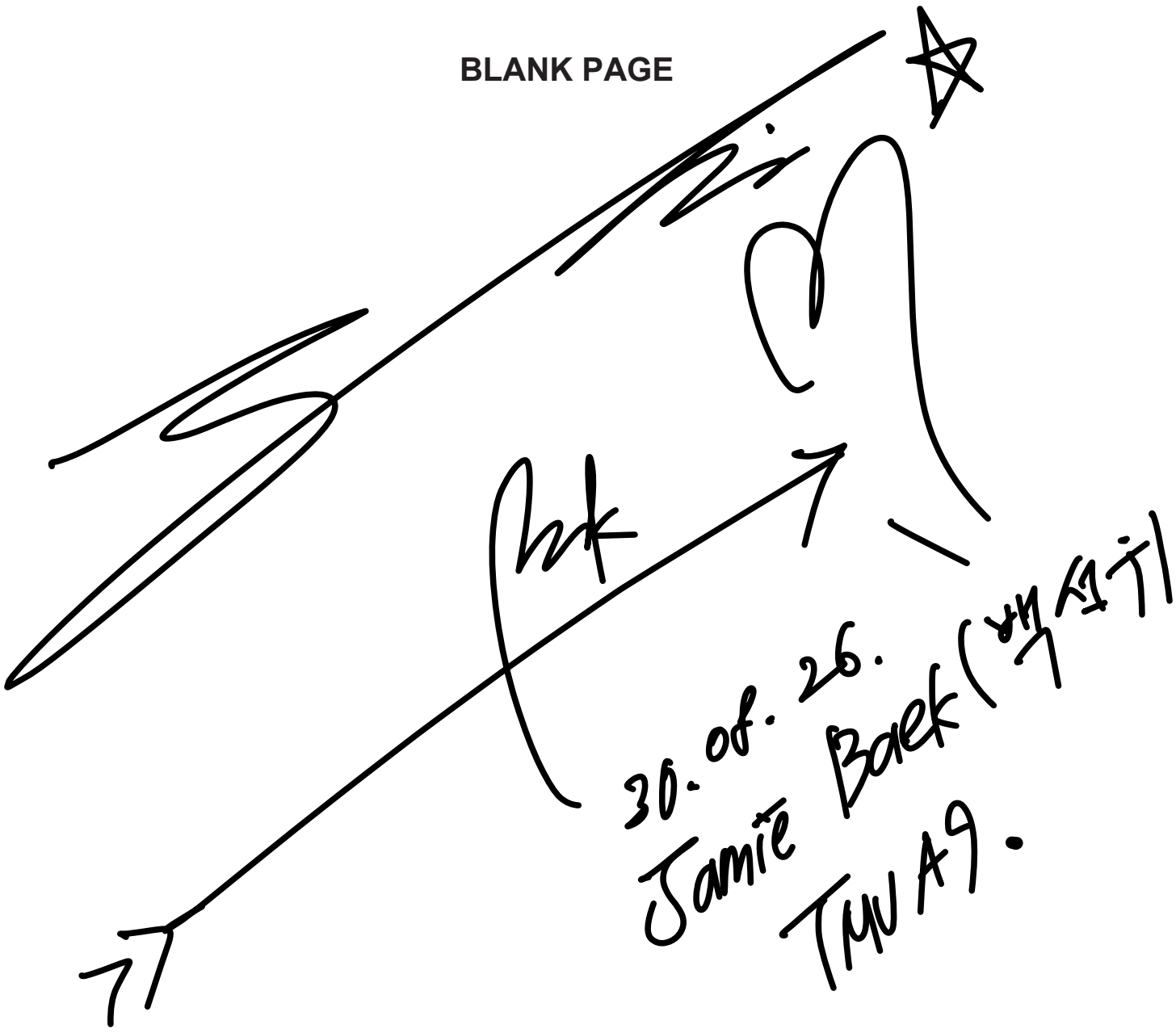


1 Find the value of

$$\int_1^4 \frac{3-2x}{x\sqrt{x}}\,\mathrm{d}x$$

A $-\frac{13}{2}$

B $-\frac{85}{16}$

C $-\frac{13}{8}$

D -1

E $-\frac{1}{4}$

F $\frac{7}{4}$

G 7

$$\int_1^4 \frac{3-2x}{x^{\frac{3}{2}}}\,dx$$

$$= \int_1^4 x^{-\frac{3}{2}}(3-2x)\,dx$$

$$= \int_1^4 \left(3x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}}\right)dx$$

Heuristic for $\int ax^n\,dx$: ① Add 1 to exponent $\longrightarrow a\int x^{(n+1)^{\star}}\,dx$. ② multiple reciprocal of $\star(=\frac{1}{\star})$ $\downarrow$ $a\times\frac{1}{\star}\int x^{\star}\,dx$. s.t. $\star = n+1$.

$$\left[3x\,x^{-\frac{1}{2}} - 2(2)x^{\frac{1}{2}}\right]_1^4$$

$$= \left[-6x^{-\frac{1}{2}} - 4x^{\frac{1}{2}}\right]_1^4$$

$$= \left(-6\times\frac{1}{2} - 4\times 2\right) - (-6-4)$$

$$= (-3-8) + 10 = -1$$

$2a_n = a_{n-k} + a_{n+k}$

view S_n as a quadratic.

2 An arithmetic progression has first term a and common difference d.

The sum of the first 5 terms is equal to the sum of the first 8 terms.

Which one of the following expresses the relationship between a and d?

A $a = -\frac{38}{3}d$

B $a = -7d$

C $a = -6d$

D $a = 6d$

E $a = 7d$

F $a = \frac{38}{3}d$

Sol 9.)

$S_5 = S_8$

Sol X) $S_n = \frac{2a + (n-1)d}{2} \cdot n$

$S_5 = S_8$

$\frac{2a+4d}{2} \times 5 = \frac{2a+7d}{2} \times 8$

Sol(1) $S_8 - S_5 = 0$

$a_8 + a_7 + a_6 = 0$

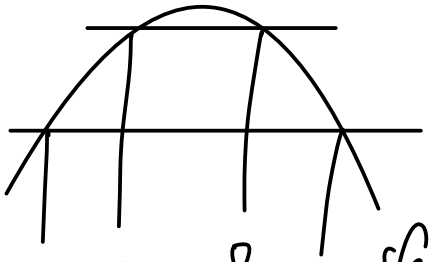
As $a_6 + a_8 = 2a_7$.

LHS: $3a_7 = 0$, so

$a_7 = 0$

$\therefore a + 6d = 0$

Sol(2) $S_8 = S_5$



$\therefore S_6 = S_7$

$\therefore S_7 - S_6 = a_7 = 0$

$\therefore a + 6d = 0$

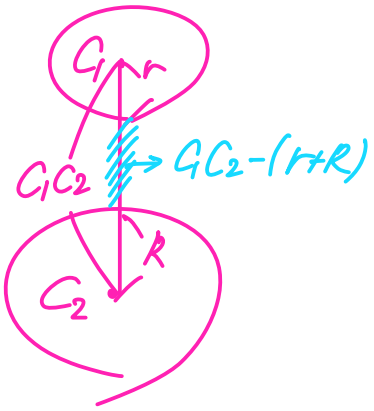
* We don't know concavity of S_n, all we know & all we need to know is whether its concave up/down.

[1] min(d) between (Circle 1) and (Circle 2) : (Distance between centre)
☆ Don't perfectly draw circles in relation to the xy coordinates. − (Sum of Radius)

3 Find the shortest distance between the two circles with equations:

$$(x+2)^2 + (y-3)^2 = 18$$

$$(x-7)^2 + (y+6)^2 = 2$$



[2-1] ✓ (−2, 3) ~ (7, −6) Distance = $9\sqrt{2}$.

[2-2] ✓ Sum of Radius = $3\sqrt{2} + \sqrt{2} = 4\sqrt{2}$

A 0

B 4

C 16

D $2\sqrt{2}$

E $5\sqrt{2}$

[3] Distance between radius: $9\sqrt{2} - 4\sqrt{2} = 5\sqrt{2}$

4 Consider the simultaneous equations

$$3x^2 + 2xy = 4$$
$$x + y = a$$

Substitute.

where a is a real constant.

Find the complete set of values of a for which the equations have two distinct real solutions for x.

A There are no values of a.

B $-2 < a < 2$

C $-1 < a < 1$

D $a = 0$

E $a < -1$ or $a > 1$

F $a < -2$ or $a > 2$

G All real values of a

$y = a - x$

$3x^2 + 2xy - 4 = 0$

$3x^2 + 2x(a - x) - 4 = 0$

$x^2 + 2ax - 4 = 0$

$(x + a)^2 = a^2 + 4$

As $a^2 + 4 > 0$,

all roots exist!

☆ factor theorem.

5 The function f is defined by $f(x) = x^3 + ax^2 + bx + c$.

a, b and c take the values 1, 2 and 3 with no two of them being equal and not necessarily in this order.

The remainder when $f(x)$ is divided by $(x + 2)$ is R.

The remainder when $f(x)$ is divided by $(x + 3)$ is S.

What is the largest possible value of $R - S$?

A -26

B 5

C 7

D 17

E 29

$R = f(-2)$

$S = f(-3)$

$R - S = f(-2) - f(-3)$

$= (-8 + 4a - 26 + c) - (-27 + 9a - 36 + c)$

$= -8 + 4a - 26 + c + 27 - 9a + 36 - c$

$= -5a + 6 + 19$

$\max(-5a + 6 + 19)$ at $\min(a)$ / $\max(b)$.

$-5(1) + (3) + 19$

$= 22 - 5 = 17$.

£ Bend to your order. (whilst being aware of errors)
give question.

6 Find the number of solutions of the equation

$$x \sin 2x = \cos 2x$$

with $0 \leq x \leq 2\pi$.

A 0

B 1

C 2

D 3

E 4

{ Never skip (1) of checking if it is/isn't zero.

Sol 1) (1) $x \cos 2x = 0$ (≡ either $x=0$, or $\cos 2x = 0$)

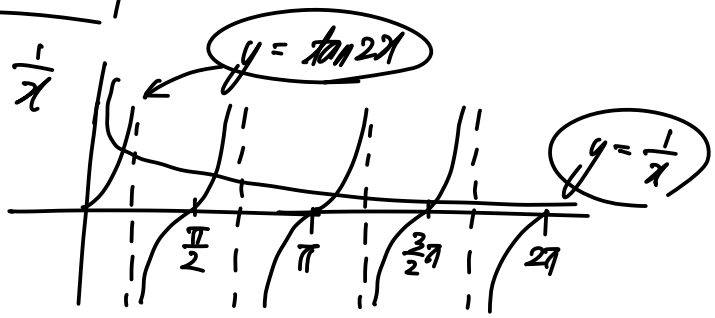
① if $x=0 \rightarrow LHS=0$, $RHS = 1 \therefore x=0$ isn't valid root.

② if $\cos 2x = 0 \rightarrow LHS \neq 0$ (non zero of $x \times \pm 1$), $RHS = 0$.
do not valid.

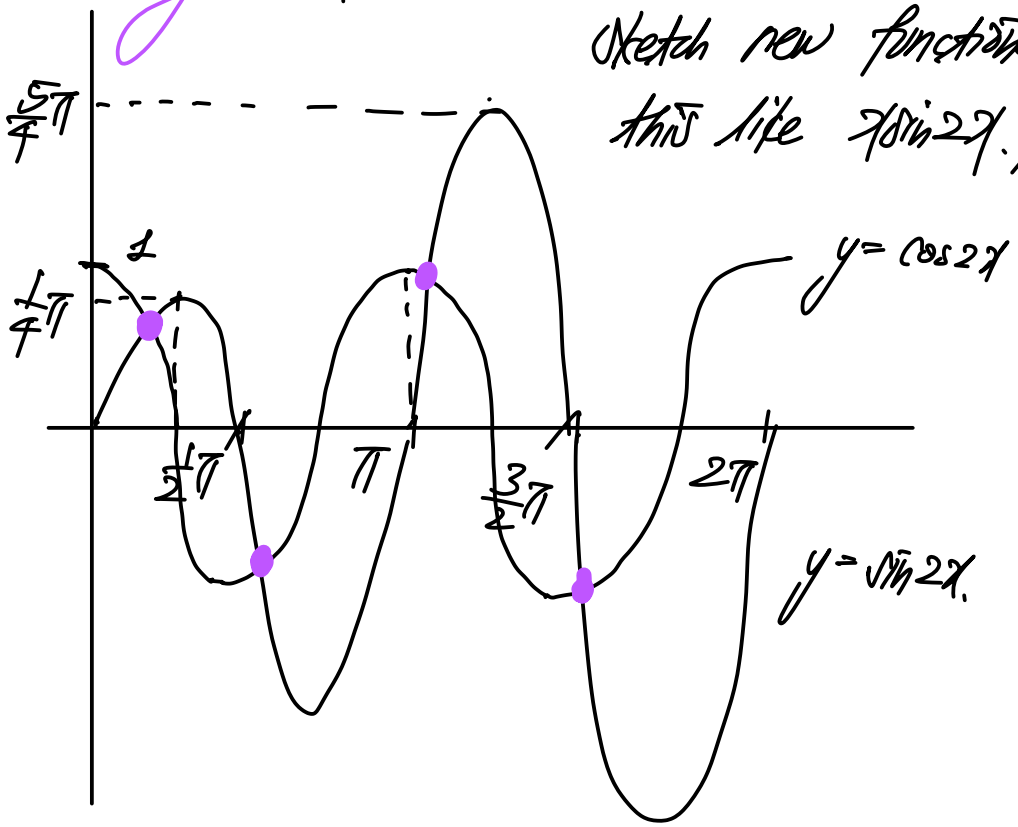
(ii) $x \cos 2x \neq 0$. (≡ both x and $\cos 2x$ isn't 0).

$\frac{\sin 2x}{\cos 2x} = \frac{1}{x}$

$= \tan 2x$.



Sol 2) Shoutout Jay K! (Y'all still must know how to sketch new functions like this like $x \sin 2x$.)



☆ Simple binomial Expansion.

☆ $nC_r = nC_{n-r}$

7 The non-zero constant k is chosen so that the coefficients of x^6 in the expansions of $(1 + kx^2)^7$ and $(k + x)^{10}$ are equal.

What is the value of k?

A $\frac{1}{6}$

B 6

C $\frac{\sqrt{6}}{6}$

D $\sqrt{6}$

E $\frac{\sqrt{30}}{30}$

F $\sqrt{30}$

$(kx^2)^3 \cdot 1^4 \times {}^7C_3 = (k)^4 (x)^6 \cdot {}^{10}C_4$

$k^3 \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = k^4 \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} \times x^6$

$35 = 210k$

$k = \frac{1}{6}$

£ Difference of squares.

£ Attention to relation.

8 The sum to infinity of a geometric progression is 6.

The sum to infinity of the squares of each term in the progression is 12.

Find the sum to infinity of the cubes of each term in the progression.

A 8

B 18

C 24

D $\frac{216}{7}$

E 72

F 216

$\frac{a}{1-r} = 6 \cdots ① \rightarrow a = 6-6r$

$\frac{a^2}{1-r^2} = 12 \cdots ②$

$(1+r)(1+r)$

$\frac{①}{②} : \frac{a}{1-r} \times \frac{1-r^2}{a^2} = \frac{6}{12}$

$\frac{1+r}{a} = \frac{1}{2} \therefore a = 2+2r$

$a = 6-6r = 2+2r$

$4 = 8r$

$r = \frac{1}{2}, a = 3.$

$\therefore \frac{a^3}{1-r^3} = \frac{27}{1-\frac{1}{8}}$

$= \frac{216}{7}$

Sol 2) Shoutout Yasin! (nearly similar, personally think this is better than my solution.)

$\frac{a}{1-r} = 6 \cdots ①$

$\frac{a^2}{1-r^2} = 12 \cdots ②$

$\hookrightarrow \underbrace{\frac{a}{1-r}}_{=6} \times \underbrace{\frac{a}{1+r}}_{\therefore =2} = 12$ → then the rest is same.

If when $h(x) = 0$ is given, split $h(x)$ into $f(x) - g(x)$ and morph into $f(x) = g(x)$ format to compare intersections.

9 Find the complete set of values of the constant c for which the cubic equation

$$2x^3 - 3x^2 - 12x + c = 0$$

has three distinct real solutions.

Critical thought

A $-20 < c < 7$

B $-7 < c < 20$

C $c > 7$

D $c > -7$

E $c < 20$

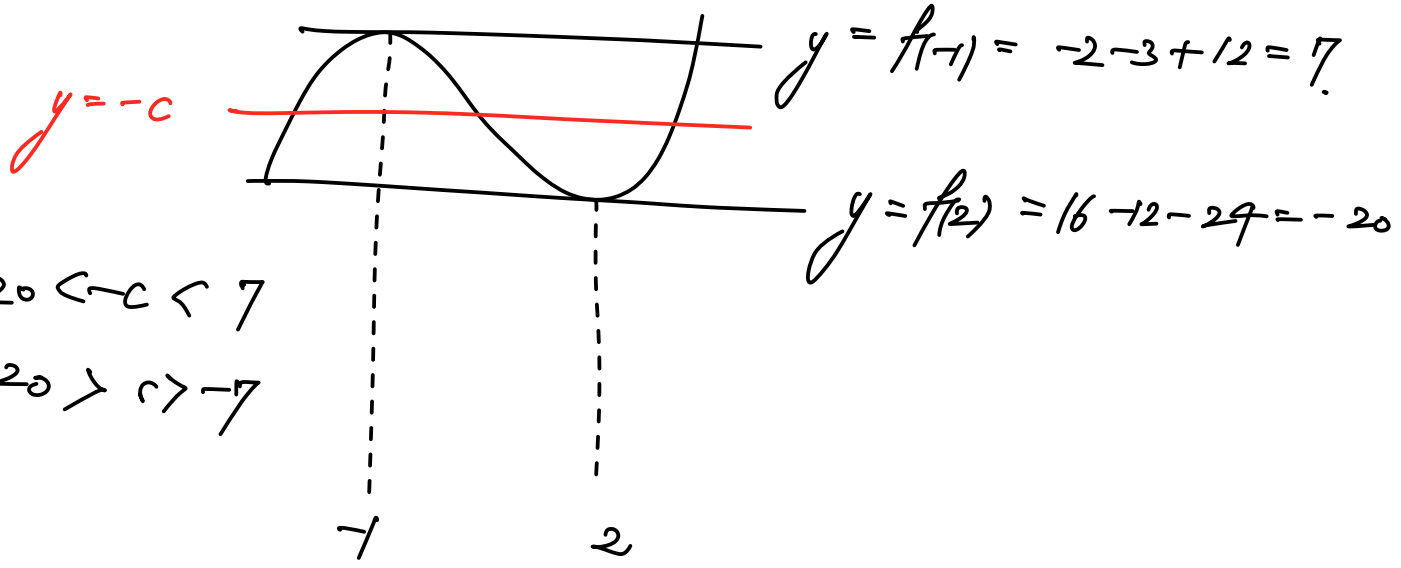
F $c < -20$

$2x^3 - 3x^2 - 12x = -c$

let $f(x) =$

$f'(x) = 6x^2 - 6x - 12$

$= 6(x-2)(x+1)$



$-20 < -c < 7$

$20 > c > -7$

£ One of the rare cases s.t. visualisation is not needed.

Part of being good is using graphs for when most use algebra but for this question visualisation is not ideal.

10 x and y satisfy $|2 - x| \leq 6$ and $|y + 2| \leq 4$.

What is the greatest possible value of $|xy|$?

E.

A 16

B 24

C 32

D 40

E 48

F There is no greatest possible value.

(Not ideal) $-6 \leq 2-x \leq 6$

$-4 \leq x \leq 8$

$-4 \leq y+2 \leq 4$

$-6 \leq y \leq 2$.

$|xy| = |x||y|$, $\max|xy| = \max|x| \times \max|y|$

$= 8 \times 6 = 48$

£ Normal to point $(p, f(p))$ on function $y = f(x)$ is

$y = -\frac{1}{f'(p)}(x-p) + f(p)$

£ Don't adhere / confine your solution on how the question presents itself. m & q is not needed at all.

11 The line $y = mx + 5$, where $m > 0$, is normal to the curve $y = 10 - x^2$ at the point (p, q).

What is the value of p?

$y' = -2x$

A $\frac{\sqrt{2}}{6}$

B $-\frac{\sqrt{2}}{6}$

C $\frac{3\sqrt{2}}{2}$

D $-\frac{3\sqrt{2}}{2}$

E $\sqrt{5}$

F $-\sqrt{5}$

Sol Ideal)

$y = -\frac{1}{-2p}(x-p) + 10 - p^2$

$= \frac{1}{2p}(x-p) + 10 - p^2$

$= \frac{1}{2p}x + 9.5 - p^2 = 5$

$p^2 = 4.5$

$\therefore p = \frac{3}{\sqrt{2}} = \frac{3}{2}\sqrt{2}$

cf) in mathematical terms this is $\int |f(x)|\,dx$

12 A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x-p)(x-q)(r-x)$$

with $0 < p < q < r$.

You are given that:

$$\int_0^r \mathrm{f}(x)\,\mathrm{d}x = 0 \qquad A - B + C = 0$$

$$\int_0^q \mathrm{f}(x)\,\mathrm{d}x = -2 \qquad A - B = -2$$

$$\int_p^r \mathrm{f}(x)\,\mathrm{d}x = -3 \qquad -B + C = -3$$

What is the total area enclosed by the curve and the x-axis for $0 \leq x \leq r$?

$\hookrightarrow A + B + C = 3 + 5 + 2 = 10$

A 0

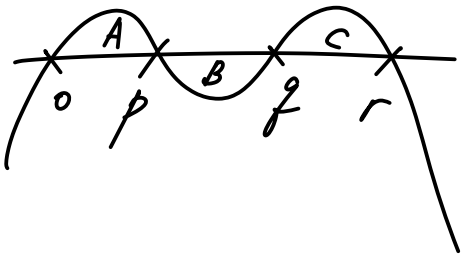
B 1

C 4

D 5

E 6

F 10



£ You should know how to sketch $y = f(x)$ from $f'(x)$ to near precision.

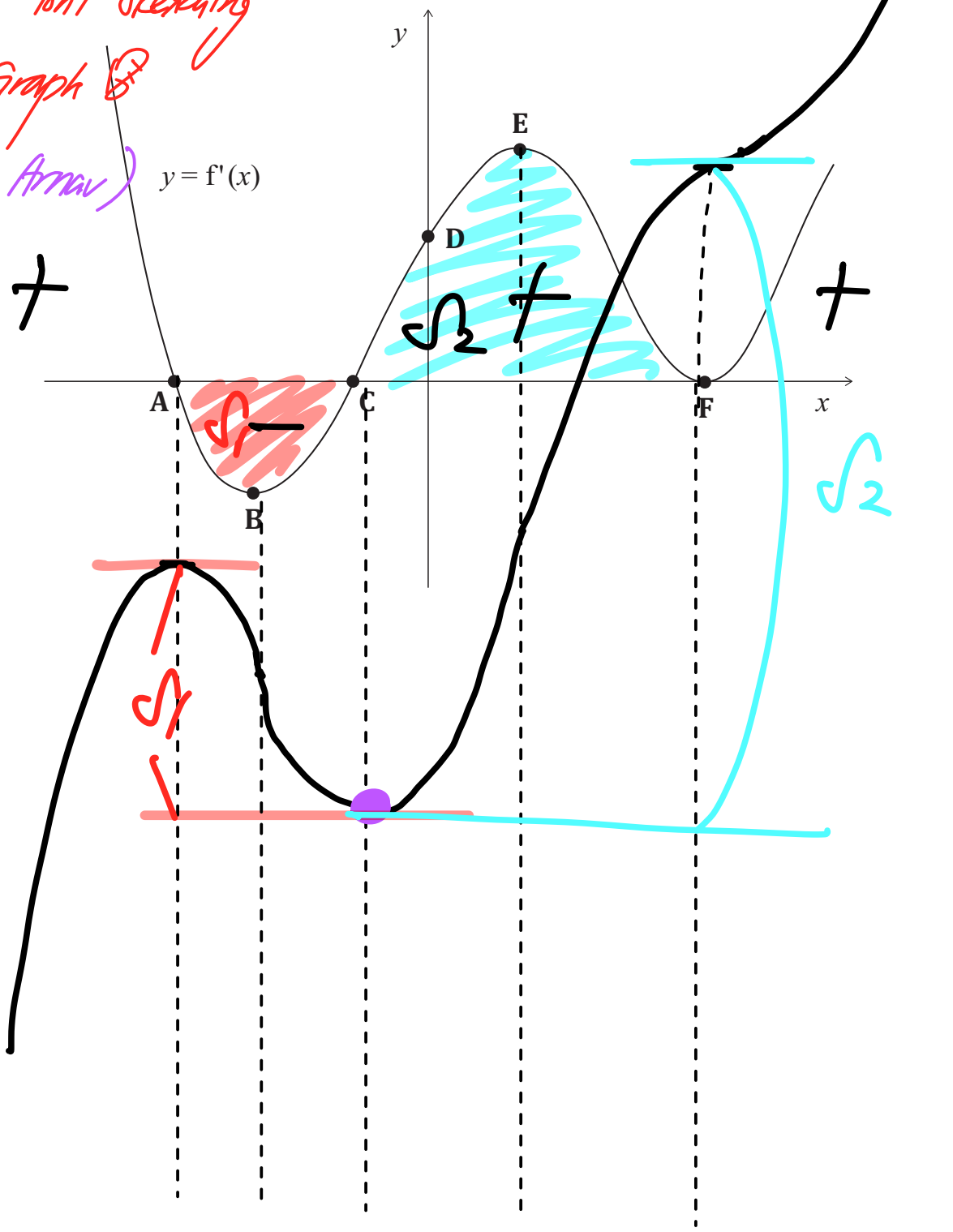
$y = f(x)$

13 The function $f(x)$ has derivative $f'(x)$.

The diagram below shows the graph of $y = f'(x)$.

Which point corresponds to a local minimum of $f(x)$?

Sol Ideal isn't sketching entire Graph 😉 (Shoutout Arnav)



14 The line $y = mx + 4$ passes through the points $(3, \log_2 p)$ and $(\log_2 p, 4)$.

What are the possible values of p?

A $p = 1$ and $p = 4$

B $p = 1$ and $p = 16$

C $p = \frac{1}{4}$ and $p = 4$

D $p = \frac{1}{4}$ and $p = 64$

E $p = \frac{1}{64}$ and $p = 4$

F $p = \frac{1}{64}$ and $p = 16$

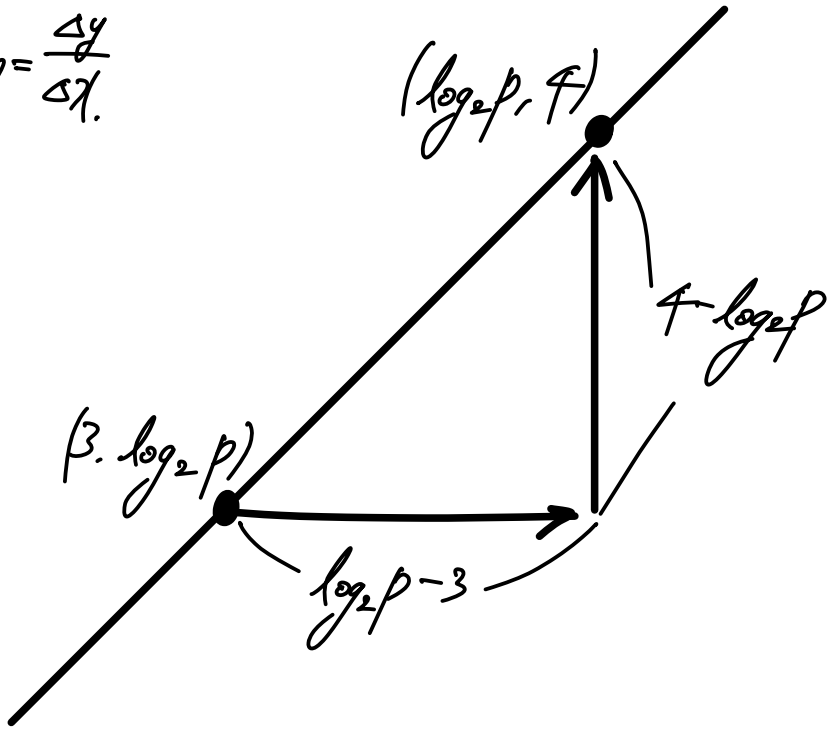
(i) $\log_2 p = 0$. $f(4)$ at $x = 0$ ($\because x = \log_2 p$)

$p = 1$

(ii) $\log_2 p = 4$. ($y = mx + 4$ just is horizontal line of $y = 4$)

$p = 16$

cf) $m = \frac{\Delta y}{\Delta x}$



I forgot to define domain of new variables.

15 Find the sum of the real solutions of the equation:

$$3^x - (\sqrt{3})^{x+4} + 20 = 0$$

A 1

B 4

C 9

D $\log_3 20$

E $2\log_3 20$

F $4\log_3 20$

let $\sqrt{3}^x = A \quad (A > 0)$ ✓

$A^2 - 9A + 20 = 0$

$A = 4, 5$

$3^x = 16, 25$

$x = \log_3 16, \log_3 25$

$\log_3 4^2 + \log_3 5^2$

$= 2\log_3 20$

Very Easy: Complete Squares.

& Stationary Point ≡ Point s.t $f'(x)=0$

16 The curve C has equation $y = x^2 + bx + 2$, where $b \geq 0$.

Find the value of b that minimises the distance between the origin and the stationary point of the curve C.

A $b = 0$

B $b = 1$

C $b = 2$

D $b = \frac{\sqrt{6}}{2}$

E $b = \sqrt{2}$

F $b = \sqrt{6}$

$y = x^2 + bx + \frac{1}{4}b^2 - \frac{1}{4}b^2 + 2$

$= \left(x + \frac{1}{2}b\right)^2 + 2 - \frac{1}{4}b^2$

$\left(-\frac{1}{2}b,\ 2 - \frac{1}{4}b^2\right)$

$\min \sqrt{\bigstar} \Leftarrow \min \bigstar$

$\bigstar = \left(\frac{-b}{2}\right)^2 + \left(2 - \frac{1}{4}b^2\right)^2$

$= \frac{1}{4}b^2 + 4 - b^2 + \frac{1}{16}b^4$

$= \frac{1}{16}b^4 - \frac{3}{4}b^2 + 4$

$= \frac{1}{16}\left(b^4 - 12b^2 + 36 - 36\right) + 4$

$= \frac{1}{16}\left(b^2 - 6\right)^2 - \frac{9}{4} + 4$

$= \frac{1}{16}\left(b^2 - 6\right)^2 + \frac{7}{4}$

☆ I'm open to hear better solutions from y'all 😊

17 There are two sets of data: the mean of the first set is 15, and the mean of the second set is 20.

One of the pieces of data from the first set is exchanged with one of the pieces of data from the second set.

As a result, the mean of the first set of data increases from 15 to 16, and the mean of the second set of data decreases from 20 to 17.

What is the mean of the set made by combining all the data?

A $16\frac{1}{4}$

B $16\frac{1}{3}$

C $16\frac{1}{2}$

D $16\frac{2}{3}$

E $16\frac{3}{4}$

let first set A, 2nd set B respectively have a, b number of pieces/elements.

$15a - ☆ + △ = 16a$

$+ \quad 20b + ☆ - △ = 17b$

$15a + 20b = 16a + 17b$ (67 dony)

$36 = a$

→ Mean of the entire stuff: $\frac{\Sigma A + \Sigma B}{a+b}$

$= \frac{15a + 20b}{a+b}$

$= \frac{45b + 20b}{45b} = \frac{65}{4} = \frac{64+1}{4} = 16\frac{1}{4}$

☆ enlargement & translation.

9. Enlargement & Translation.

18 What is the smallest positive value of a for which the line $x = a$ is a line of symmetry of the graph of $y = \sin(2x - \frac{4\pi}{3})$?

A $\frac{\pi}{12}$

B $\frac{5\pi}{12}$

C $\frac{7\pi}{12}$

D $\frac{11\pi}{12}$

E $\frac{19\pi}{12}$

(B)

$y = \sin ☆$ has line of symmetry at $☆ = \frac{1}{2}(2n\pi)\pi$.

s.t. $n \in \mathbb{Z}$. (integer)

$2x - \frac{4}{3}\pi = -\frac{9}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \ldots$

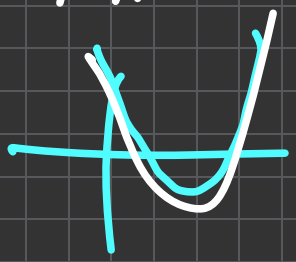
$2x = -\frac{1}{6}\pi, \frac{5}{6}\pi, \ldots$

$x = -\frac{1}{12}\pi, \frac{5}{12}\pi, \ldots$

min

9. Enlargements & Translations.

⊕ Trigonometry. (Areas of Trig).



⊕ Modulus

(Tian).

- $f(x) \to f(-x)$ → Reflections in y-axis

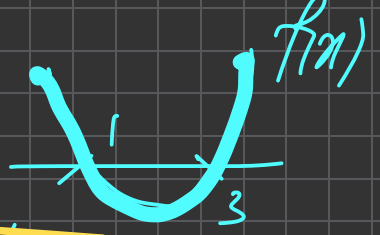
(Arnav).

- $f(x) + f(-x) = 0$. → Point symmetry abt Origin
- $f(x) \to -f(x)$ → flip on x-axis (Jakub!)

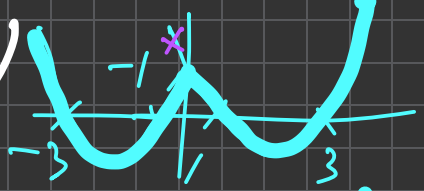
try $f(x) = x^2$

- $y = f(|x|)$ → symmetry about y-axis.
- $y = |f(x)|$ → Reflection of negative y values upwards of x-axis.
- $|y| = f(x)$ → Discard $y < 0$ region, flip 1st & 2nd quadrant by x-axis.
- $|y| = |f(x)| \to y = \pm f(x)$
- $|y| = f(|x|)$ leave 1st quadrant, truncate rest & flip.

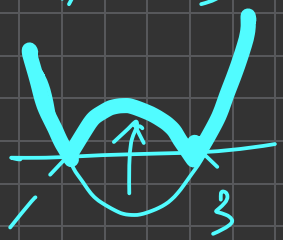
$f(x) = (x-1)(x-3)$



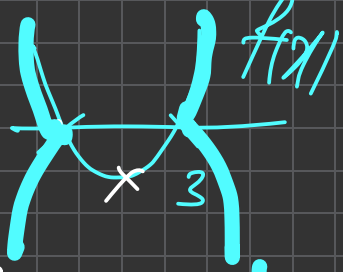
$y = f(|x|)$



$y = |f(x)|$

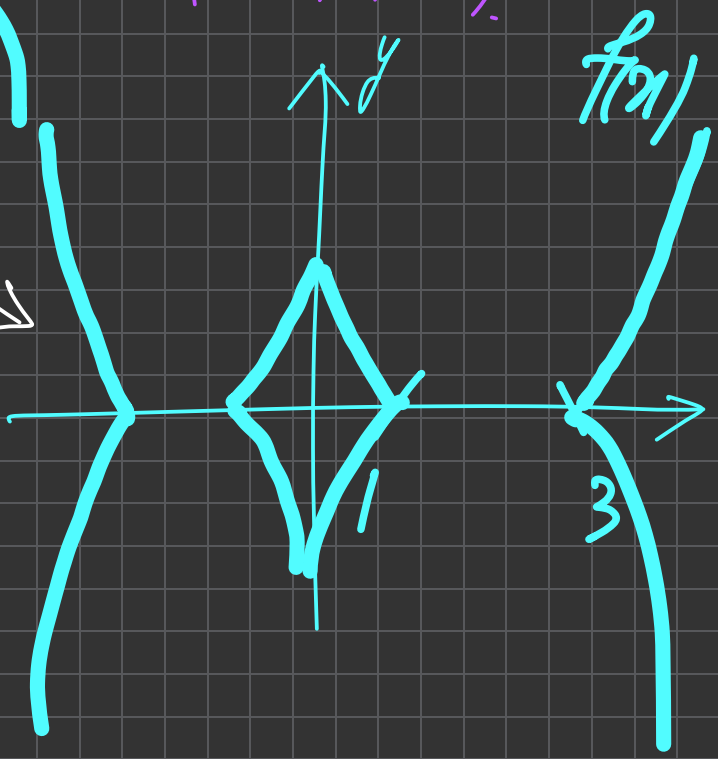


$|y| = f(x)$



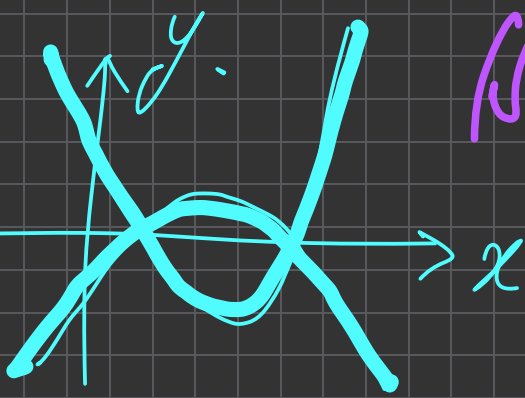
(w Alex)

$|y| = f(|x|)$



$|y| = |f(x)|$

$y = \pm f(x)$

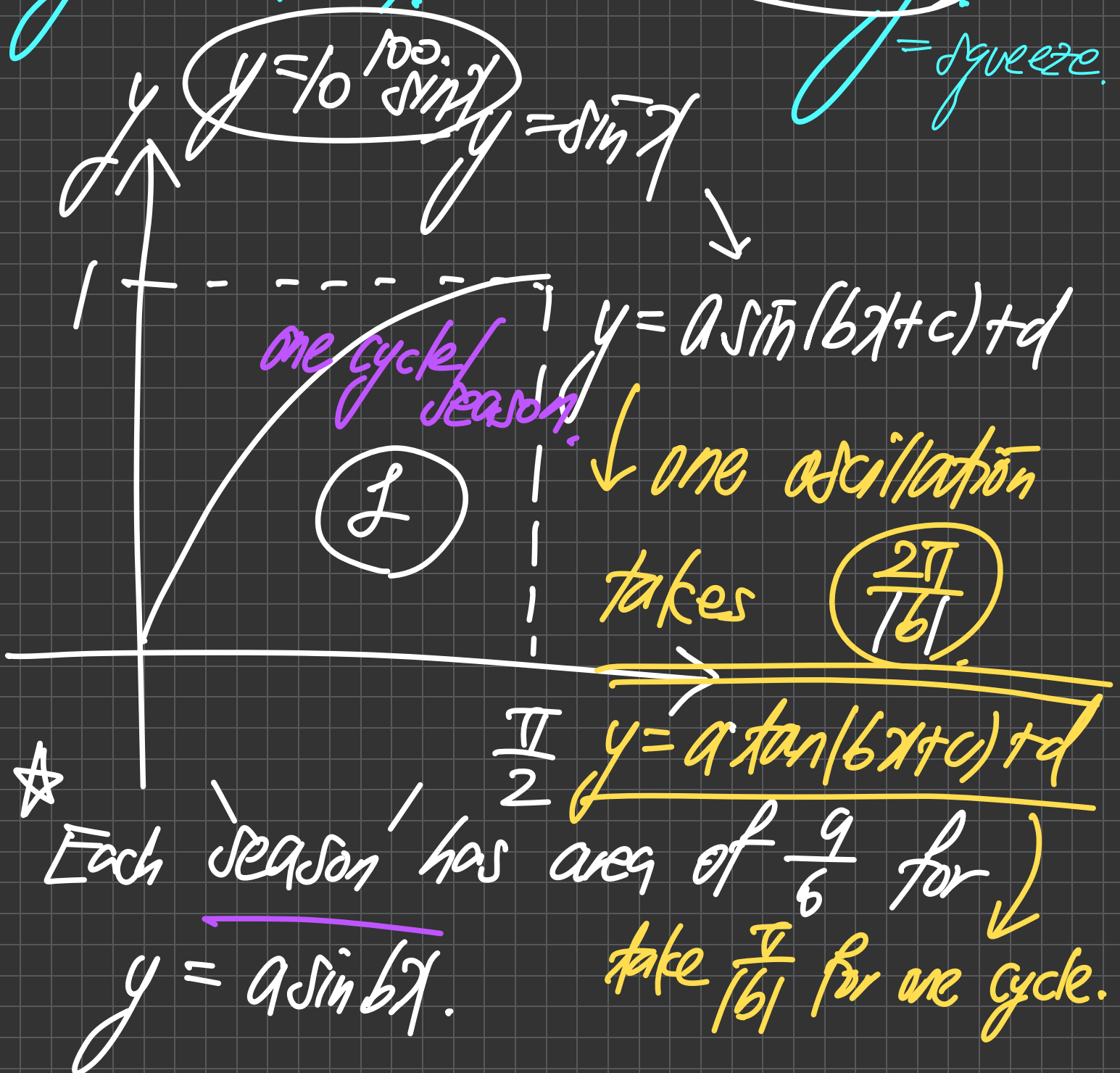


(Shoutout Tang)

$y = \sin x$ ——— normal

$y = \sin\left(\frac{1}{2}x\right)$ ——— fat = stretch

$y = \sin(2x)$ ——— skinny = squeeze.



★ Each season has area of $\frac{a}{b}$ for $y = a\sin bx$.

take $\frac{\pi}{|b|}$ for one cycle.

✓ axes
✓ line of symmetries
✓ point of symmetries
} in sty ✓

$y = (x-1)^3$: point symmetric about $(1, 0)$ ✓✓

↓ $f(\star) + f(\triangle) = 2b$ s.t. $a = 1$, $b = 0$.

$f(4x-2) + f(-4x+4) = 0$ ///

$\star + \triangle = 2a$

Point Symmetry

★ If $f(\star) + f(\triangle) = 2b$, then $f(x)$ is point symmetric about (a, b).

Line Symmetry

★ If $f(\star) = f(\triangle)$, $f(x)$ is line symmetric about a such that $\star + \triangle = 2a$.

$f(x) = (x-1)(x-3)$ ✓✓

$f(1+x) = f(3-x)$ ✓

$f(x)$ $f(4-x)$ ↙

→ line symmetry about $x = 2$.

$f(\pi + 4x) = f(4 - \pi - 4x)$

$f(x) = (x-1)^3 + 4$ (1, 4)

a b
↓ ↓
2 — (1, 4)

$f(\alpha) + f(\beta) = 26$

$\frac{\alpha+\beta}{2} = a$

P(1, 4)

$f(x) + f(2-x) = 8$ ✓

$f(1+x) + f(1-x) = 8$

$f\left(\left(\frac{1}{4}\right)^e + \pi x\right) + f\left(-\left(\frac{1}{4}\right)^e - \pi x + 2\right)$

① numbers with parentheses & coefficients of x → both irrelevant

$= 8.$

£I feel like my method was just too aura... The reason I thought of circles here was to think of a fixed point of B and same two lengths. (Definition of circle!)

19 A triangle ABC is to be drawn with $AB = 10\text{cm}$, $BC = 7\text{cm}$ and the angle at A equal to θ, where θ is a certain specified angle.

Of the two possible triangles that could be drawn, the larger triangle has three times the area of the smaller one.

What is the value of $\cos\theta$?

A $\frac{5}{7}$

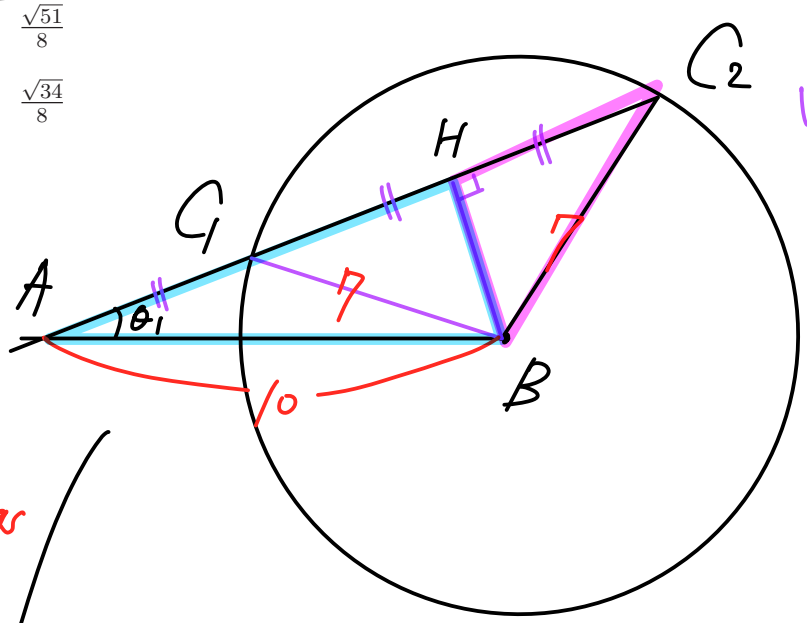
B $\frac{151}{200}$

C $\frac{2\sqrt{2}}{5}$

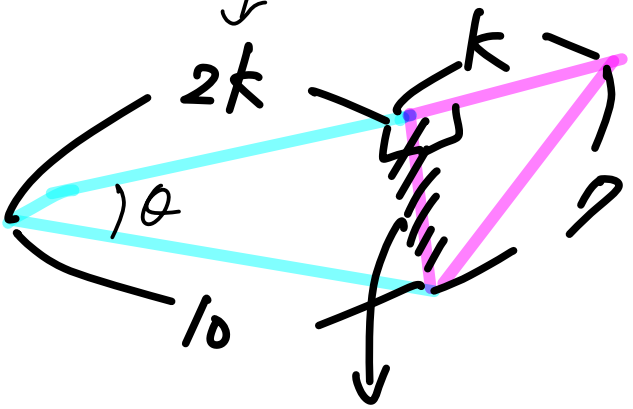
D $\frac{\sqrt{17}}{5}$

E $\frac{\sqrt{51}}{8}$

F $\frac{\sqrt{34}}{8}$



Best Analogy (Shoutout Arnav actually goated)
→ The swings of pendulums with fixed B.



Using Pythagoras

Using pythagoras: $10^2 - (2k)^2 = 7^2 - k^2$

$51 = 3k^2$

$k^2 = 17 \quad k = \sqrt{17}$

$\cos\theta = \frac{2k}{10} = \frac{k}{5} = \frac{\sqrt{17}}{5}$

☆ $\sin\theta = \cos(90-\theta)$, $\sin(90-\theta) = \cos\theta$.

$\therefore \sin^2\theta + \sin^2(90-\theta) = \sin^2\theta + \cos^2\theta = 1$.

20 Find the value of

$$\sin^2 0° + \sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 87° + \sin^2 88° + \sin^2 89° + \sin^2 90°$$

E

A 0.5

B 1

C 1.5

D 45

E 45.5

F 46

let this be S.

$S = \sin^2 0 + \sin^2 1 + \cdots + \sin^2 44 + \sin^2 45 + \sin^2 46 + \cdots + \sin^2 90$

$S = \sin^2 90 + \sin^2 89 + \cdots + \sin^2 46 + \sin^2 45 + \sin^2 44 + \cdots + \sin^2 0$

$2S = \underbrace{1+1+\cdots+1}_{91 \text{ of them } (0\sim 90)}$

$\therefore S = 45.5$

END OF TEST

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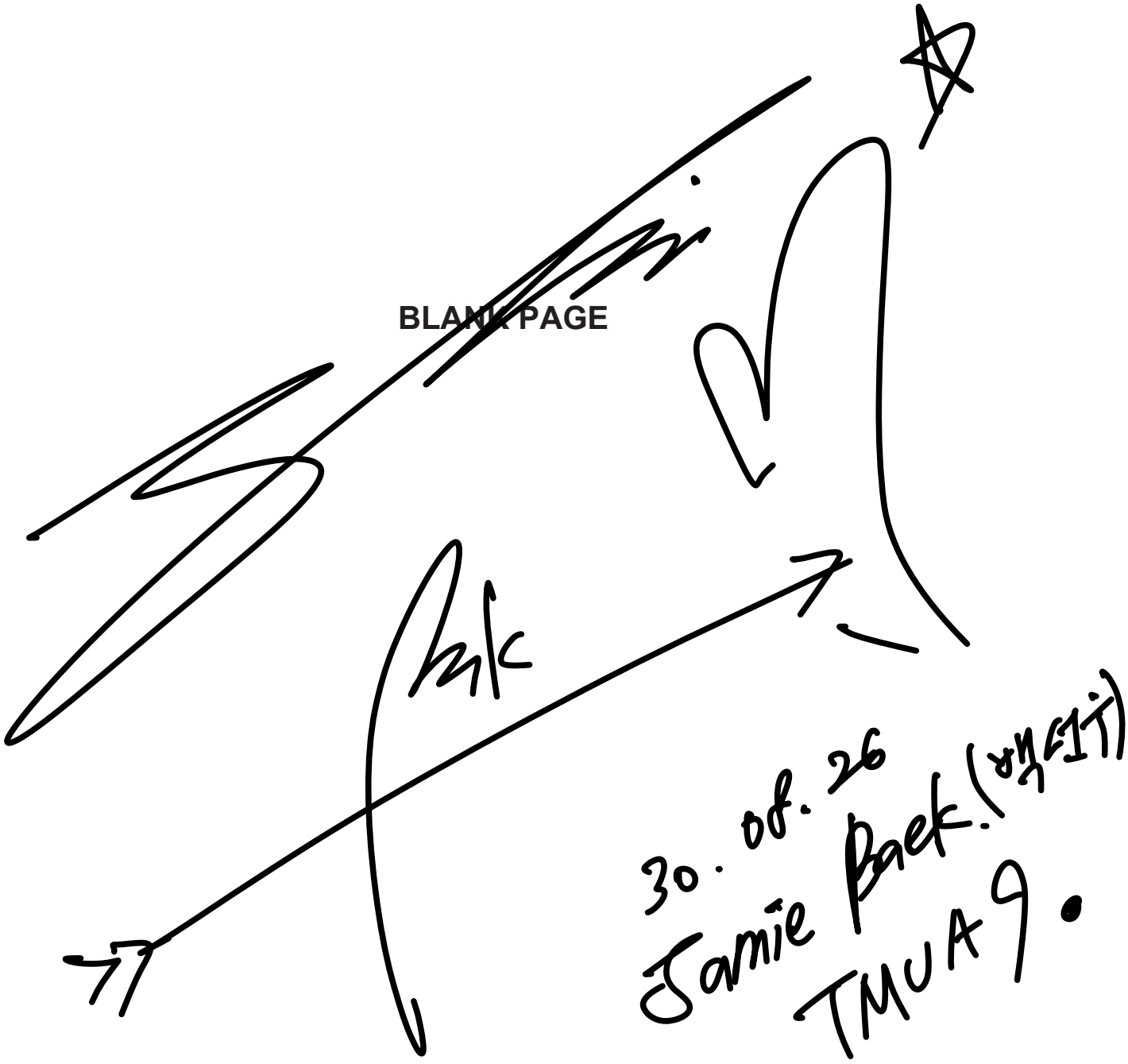

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