The Forty-Sixth Annual William Lowell Putnam Competition
Saturday, December 7, 1985

A–1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

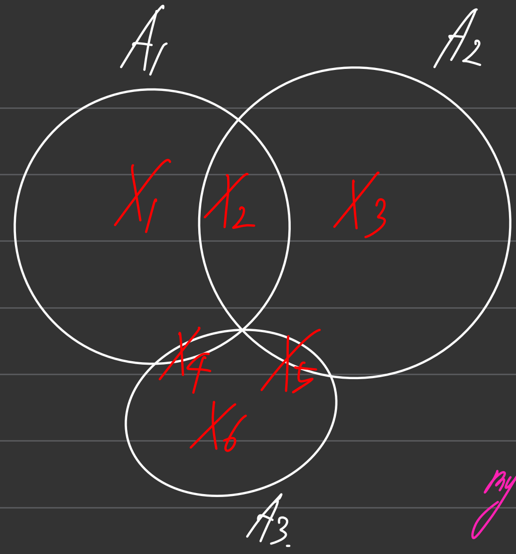
(i) $A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a,b,c,d are nonnegative integers.

A–2 Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

(A-1)

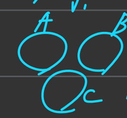


☆ Sketch Venn Diagram for visualisation.

Confluences for set theory:

* Caveats: $\{A\cap B\cap C=\varnothing\} \neq \{A\cap B=\varnothing\}$.

→ This mistake made me think Just



this 3^{10}.

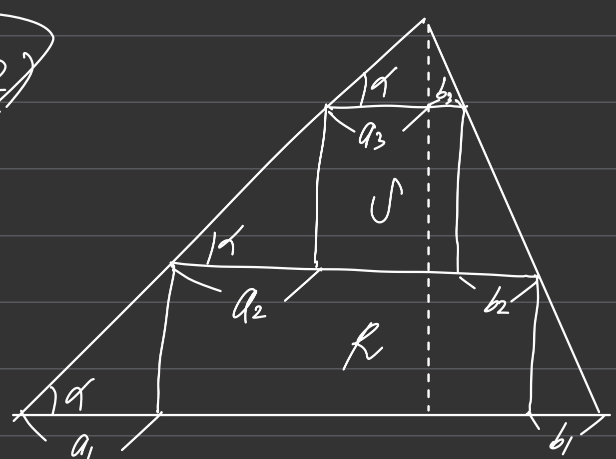
my erroneous thought process: $A\cap B\cap C=\varnothing$.

They were all discrete. (X)

$X_1 \sim X_6$ ...

· there a 6 sets that elements $n=1,2\cdots 10$ can slide into.
Hence, 6^{10} as total possible outcomes. → Total possible outcome $=2^{10}3^{10}5^{0}7^{0}$
So $(a,b,c,d)=(10,10,0,0)$.

(A-2)



☆ Sketch.

☆ for similar triangles, set variables of corresponding sides as a sequence/ a_n & set variables for sides s.t minimises total count of variables.

☆ Cauchy Schwartz formulae) : Confluences =
- inequality ($\le / \ge$)
- max/min
- Coexistence of $(x^2+y^2+z^2)$ and $(x+y+z)^2$.

$\left\{\begin{array}{l} x^2+y^2+z^2 \\ (x+y+z)^2 \\ xy+yz+zx \end{array}\right.$ co-adjacent concepts

$$a_k\tan\alpha = b_k\tan\beta$$

$$a_k b_k\tan\alpha = b_k^2\tan\beta$$

$$A(T)=A(S)+A(R)+\Sigma\, 6 \text{ triangles}.$$

$$\frac{A(R)+A(S)}{A(T)} = \frac{A(T)-\Sigma\, 6\text{ triangles}}{A(T)} = 1-\frac{\Sigma\, 6 \text{ triangles}}{A(T)}$$

$$=1-\frac{\cancel{\frac{1}{2}}\sum_{k=1}^{3}a_k^2\tan\alpha+\cancel{\frac{1}{2}}\sum_{k=1}^{3}b_k^2\tan\beta}{\left(\sum_{k=1}^{3}a_k+\sum_{k=1}^{3}b_k\right)\times\left(\tan\alpha\cdot\sum_{k=1}^{3}a_k\right)\times\cancel{\frac{1}{2}}}$$

$$=1-\frac{\sum_{k=1}^{3}\left(a_k^2\tan\alpha+a_kb_k\tan\alpha\right)}{\left(\sum_{k=1}^{3}a_k+\sum_{k=1}^{3}b_k\right)\times\tan\alpha\sum_{k=1}^{3}a_k}$$

$$=1-\frac{\sum_{k=1}^{3}\left(a_k^2+a_kb_k\right)}{\sum_{k=1}^{3}(a_k+b_k)\times\sum_{k=1}^{3}a_k} \rightarrow$$

$$1-\frac{a_1^2+a_2^2+a_3^2+(a_1b_1+a_2b_2+a_3b_3)}{(a_1+a_2+a_3+b_1+b_2+b_3)(a_1+a_2+a_3)}$$

$$=1-\frac{a_1^2+a_2^2+a_3^2+(a_1b_1+a_2b_2+a_3b_3)}{(a_1+a_2+a_3)^2+(a_1+a_2+a_3)(b_1+b_2+b_3)}$$

As $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\frac{\tan\beta}{\tan\alpha}$.

① $\frac{a_1^2}{a_1b_1}=\frac{a_2^2}{a_2b_2}=\frac{a_3^2}{a_3b_3}=\frac{\tan\beta}{\tan\alpha}$.

thus,

$$\frac{a_1^2+a_2^2+a_3^2}{a_1b_1+a_2b_2+a_3b_3}=\frac{\tan\beta}{\tan\alpha},\quad \text{so } \sum_{k=1}^{3}a_kb_k=\frac{\tan\alpha}{\tan\beta}\sum_{k=1}^{3}a_k^2$$

② $\frac{\sum_{k=1}^{3}a_k}{\sum_{k=1}^{3}b_k}=\frac{\tan\beta}{\tan\alpha}$, $\sum_{k=1}^{3}b_k=\frac{\tan\alpha}{\tan\beta}\times\sum_{k=1}^{3}a_k$

Denominator: $\sum_{k=1}^{3}(a_k+b_k)\times\sum_{k=1}^{3}a_k=\sum_{k=1}^{3}a_k\left(1+\frac{\tan\alpha}{\tan\beta}\right)\times\sum_{k=1}^{3}a_k$

$$=\left(\sum_{k=1}^{3}a_k\right)^2\left(1+\frac{\tan\alpha}{\tan\beta}\right)$$

$$1-\frac{\sum_{k=1}^{3}a_k^2+\sum_{k=1}^{3}a_kb_k}{\sum_{k=1}^{3}(a_k+b_k)+\sum_{k=1}^{3}a_k}=1-\frac{\sum_{k=1}^{3}a_k^2\left(1+\frac{\tan\beta}{\tan\alpha}\right)}{\left(\sum_{k=1}^{3}a_k\right)^2\left(1+\frac{\tan\beta}{\tan\alpha}\right)}$$

$$=1-\frac{\sum_{k=1}^{3}(a_k)^2}{\left(\sum_{k=1}^{3}a_k\right)^2}\le 1-\frac{1}{3}=\frac{2}{3}.$$

the confluences were.

≤ maximum value

$$\frac{\sum ★^2}{(\sum ★)^2}=\frac{\text{sum of square}}{\text{square of sums}}.$$

$$\therefore \frac{A(R)+A(S)}{A(T)}\le\frac{2}{3}$$

★ Cauchy Schwartz's Theorem

why?

$\sum_{k=1}^{n}a_k^2$ & $\left(\sum_{k=1}^{n}a_k\right)^2$

$(a_1^2+a_2^2+\cdots+a_n^2)(1^2\times n)\ge(a_1+a_2+\cdots+a_n)^2$.

Generalized Cauchy. $n\sum_{k=1}^{n}a_k^2\ge\left(\sum_{k=1}^{n}a_k\right)^2$. General form of Cauchy.

★ There are "key thoughts" that are master keys that open doors!

★ Reverse Engineer the confluences of entry signals

→ Retrace & Reverse engineer the 'why'

→ Extract principles and leverage them via backtesting.

★ Thales' Theorem / Homogeneity of Linear Transformations Lemma.

why?

from sketch: $a_k\tan\alpha=b_k\tan\beta$.

↳ $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\frac{\tan\beta}{\tan\alpha}$.

★★ if $\frac{a_k}{b_k}=c$ (C = Real Constant),

then $\frac{\sum a_kn_k}{\sum b_kn_k}=C$.

$\frac{\sum ○\times□}{\sum \triangle\times\nabla}$ constant

★ Confluence: Similar structure multiplied amongst the maths expressions.

writing this was critical.

★ Scale surface-area of P(revelation) by jotting down all variances of all formulae that show up!

sum of product. / sum of product.

linear algebra. conservation of properties!