TMUA9. ESPRESSO.

Polynomials.

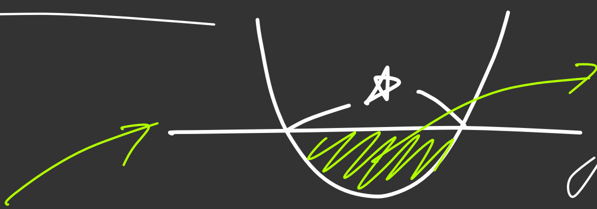
The Caveat worth noting:

If you fully master all formula beyond mere memorization. then it will cut TMUA exam time by orders of magnitude. But not doing so, or only vaguely knowing formulae has catastrophic repercussions. Even if you know 90% of a formula, you are worse off than someone who doesn't, because it just complexifies the situation.

It's a warning that if you're going to risk confusion go all in.
I personally promise you it will save you time greatly. Let's go.

£ formula £

$f(x) = ax^2+bx+c \ (a \neq 0)$

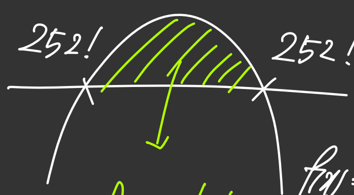


$S_{☆} = \frac{1}{6}|a|☆^3.$

$y = k \ (k \in \mathbb{R})$

- Let $f(x) = ax^2+bx+c$, ($a \neq 0$, a, b, c are real numbers.) and $y=k$ (k is also real constant).

£ formula £ Epilogue:



$y = e^{982} + \pi^{-2}$

$f(x) = -\frac{6}{512}x^2 + \sim$

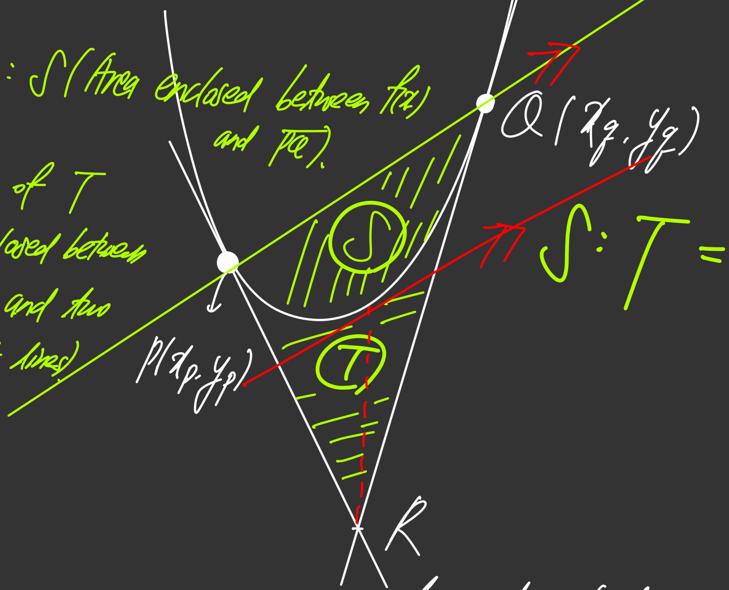
$S_{m} = \frac{1}{6}|a|☆^3$

$= \frac{1}{6}\left|-\frac{6}{512}\right| \times (252!+8-252!)^3 = 1.$

Quadratic ⇒ two tangents ⇒ intersection. $= (R)$
at two points (P, Q)

$y = f(x) = ax^2 + \sim \; (a \neq 0,\ a \in \mathbb{R}).$
so $f(x)$ is a quadratic ∈ Parabola.

Formula: S (Area enclosed between $f(x)$ and $\overline{PQ}$).
is twice of T (area enclosed between $y = f(x)$ and two tangent lines)



$S : T = 2 : 1$

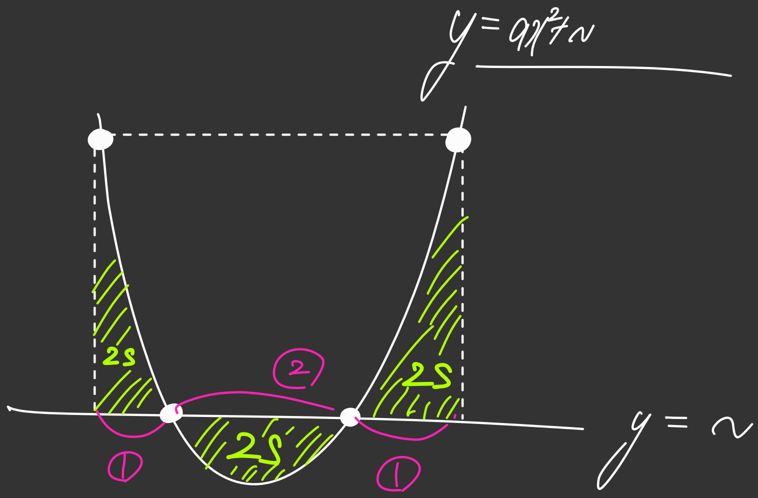
l_Q l_P = tangent to $y = f(x)$ at point P
↳ tangent to $y = f(x)$ at point Q

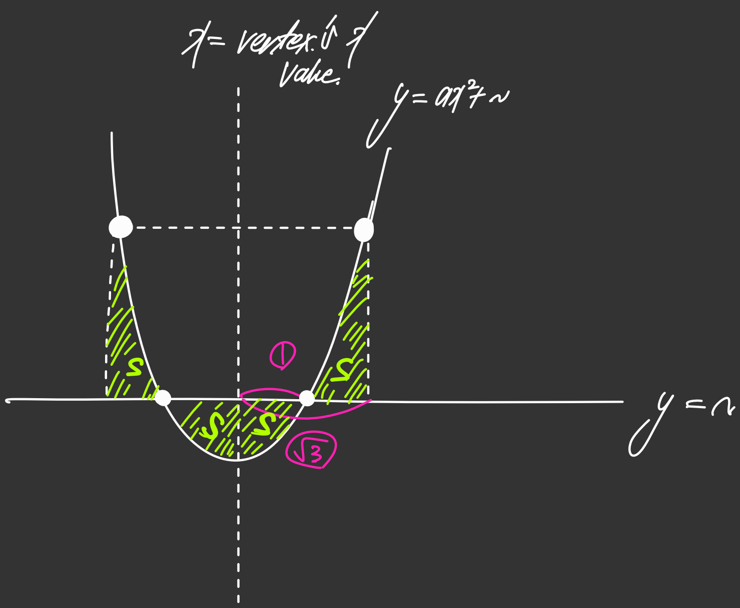
⊕ let $R(x_R, y_R)$:

Attributes
$$\begin{cases} x_R = \frac{1}{2}(x_P + x_Q). \\ f'(x_R) = \dfrac{y_Q - y_P}{x_Q - x_P}. \end{cases}$$

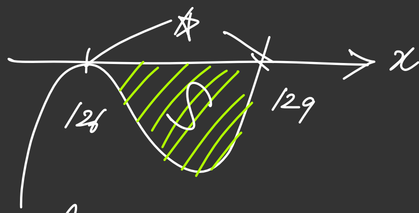
→ instantaneous rate of $f(x)$ at $x = x_R$ / gradient of tangent to $y = f(x)$ at $x = x_R$

Gradient of line $\overline{PQ}$

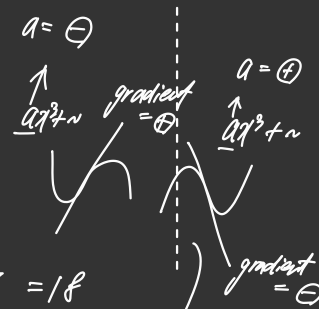

y = ax²+n
2S
2S
2S
2
1
1
y = n


x = vertex의 x
Value.
y = ax²+n
S
S
S
S
1
√3
y = n

Parable



$y=f(x)=8(x-126)^2(x-129)$



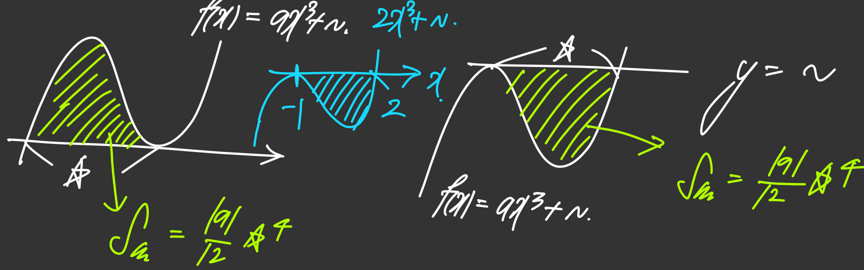
Sol(x) $S=\int_{126}^{129} 0-\left(8(x-126)^2(x-129)\right)dx$

Sol(●) $S=\frac{|8|}{12}(129-126)^3 = \frac{8}{12}\times 3^3 = \frac{2}{3}\times 27 = 18$

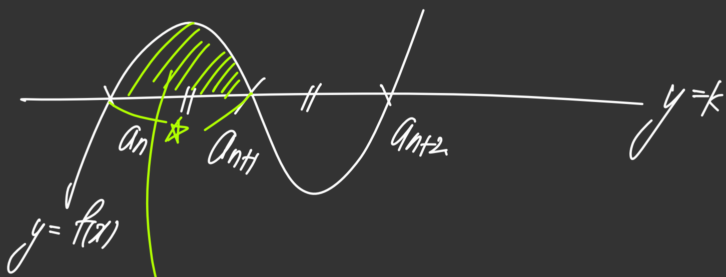
Prologue

- the cubic function must be a 'worm' shape. (A worm shape is a cubic function s.t the leading coefficient and the gradient at inflection point are opposite signs.)
- the line must be horizontal.
- the line must be tangent at local maximum or local minimum.

Formula 2. $f(x)$ meets $y=k\ (k\in\mathbb{R})$ at 2 points, one point is repeated roots.

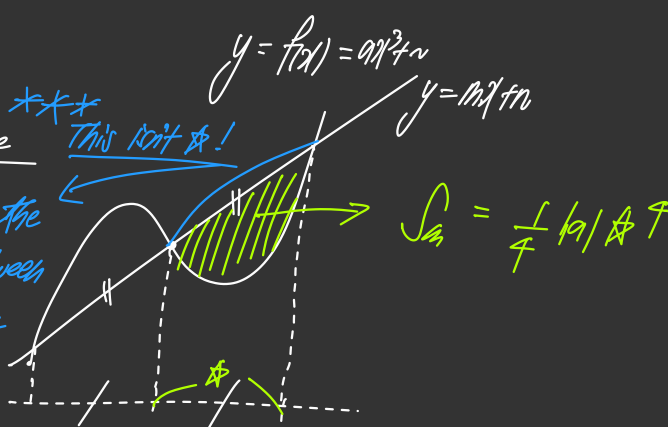


Formula 3. $f(x)$ meets $y=k\ (k\in\mathbb{R})$ at 3 points & their x coordinates are all equidistant.



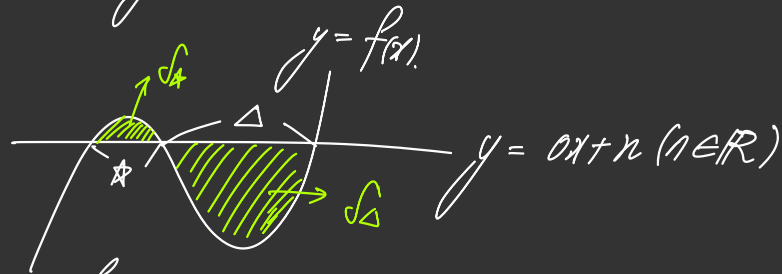
Area $S_{an}=\frac{1}{4}|a|\cdot☆^4$, $☆=a_{n+1}-a_n$.

formula 3 Epilogue ***



* ☆ here isn't the actual distance between roots itself, it's the distance between [X-coordinates]

formula 4. Area of any function that meets with a horizontal line.

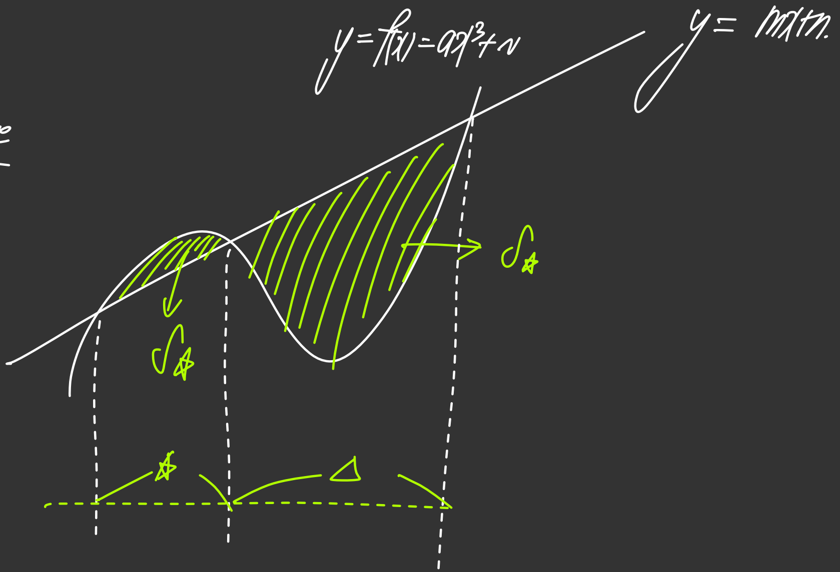


$y = f(x) = ax^3 + n.\ (a \neq 0,\ a \in \mathbb{R})$

* $S_{\star} = \frac{|a|}{6}\star^3 \times \left(\frac{1}{2}\star + \Delta\right)$

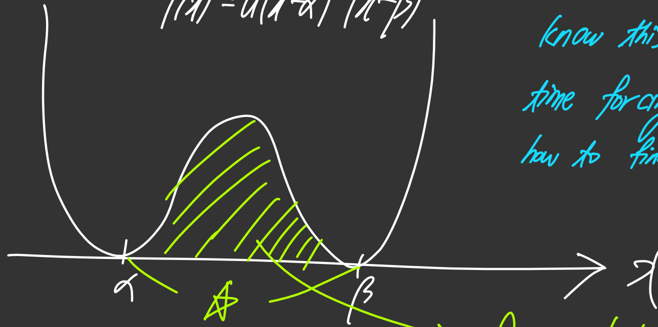
* $S_{\Delta} = \frac{|a|}{6}\Delta^3\left(\frac{1}{2}\Delta + \star\right)$

formula 4 Epilogue



£ formula 5

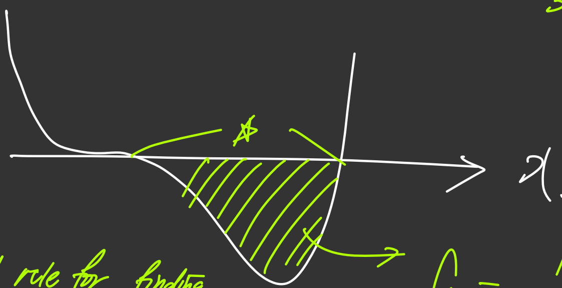
$f(x) = a(x-\alpha)^2(x-\beta)^2$



$$\int_{\star} = \frac{|a|}{30}\star^5$$

I don't think you need to know this nor should you waste time forcing yourself over obsessing how to find them if it's cumbersome.

£ formula 6



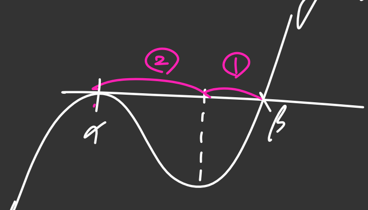
$$\int_{\star} = \frac{|a|}{20}\star^5$$

*There is a general rule for finding area enclosed between quartic and horizontal line, but it is not included as it is does not show up on TMUA/ESAT.

£ formula 7&8

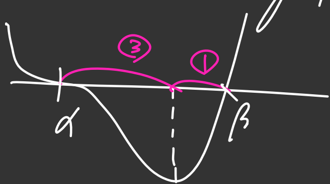
length ratios

$y = a(x-\alpha)^2(x-\beta)$



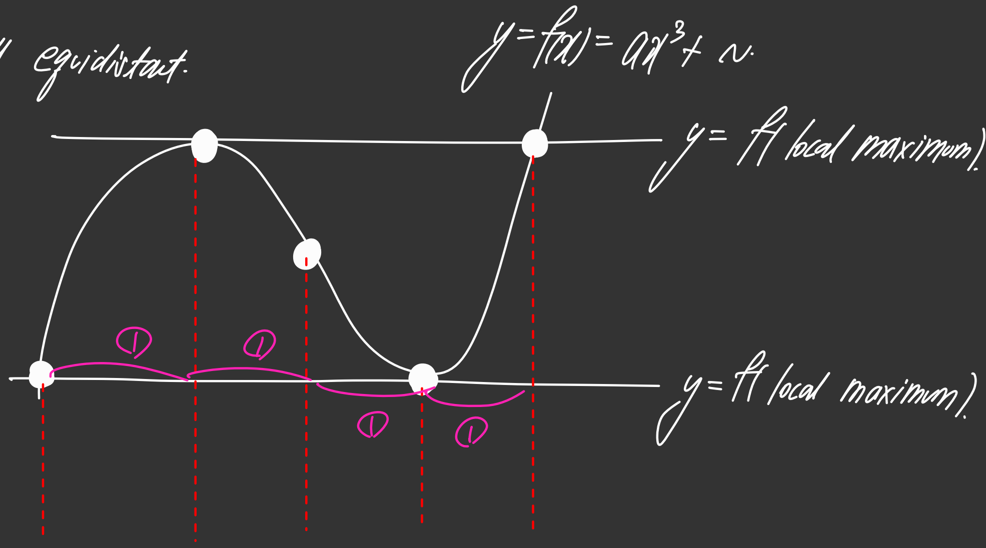
$y = \star$

$y = a(x-\alpha)^3(x-\beta)$

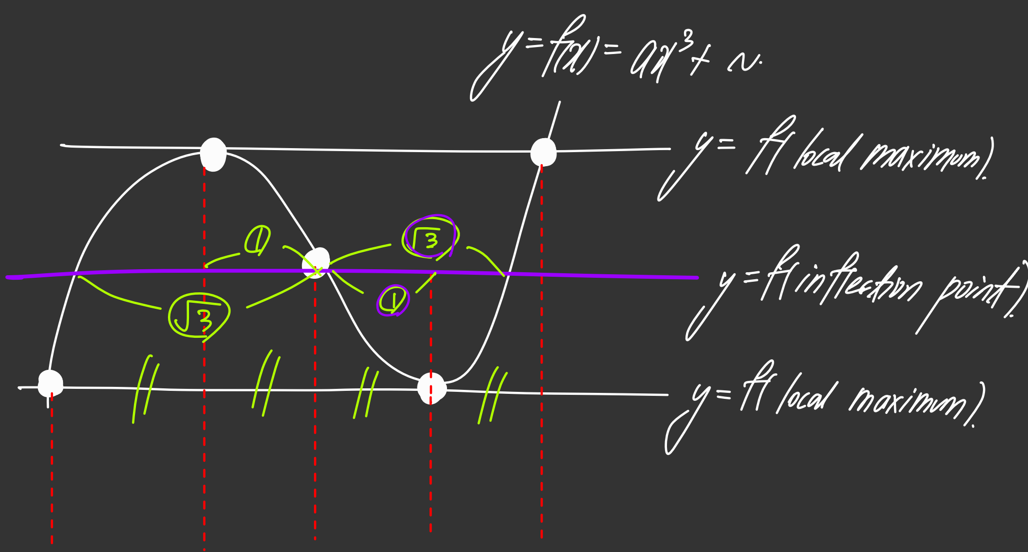


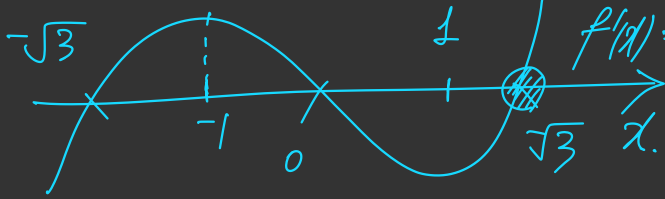
$y = \star$

£ Are all equidistant.



$y = f(x)$ has 3 intersections with the horizontal line that passes through the inflection point.



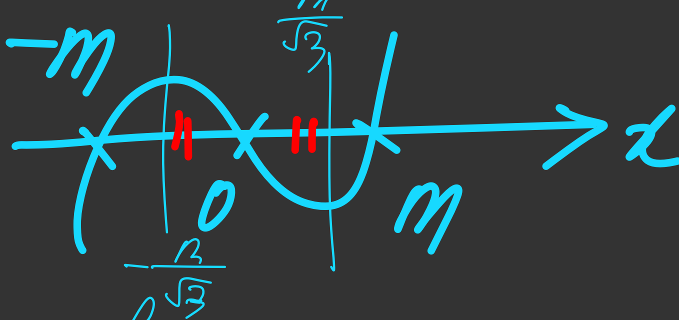


$f(x) = x^3 - 3x.$

$f'(x) = 3x^2 - 3.$

$= 3(x-1)(x+1)$

$f(x) = x(x^2-3) = x(x-\sqrt{3})(x+\sqrt{3})$



$f(x) = ax(x-m)(x+m).$

$= ax^3 - am^2x.$

$f'(x) = 3ax^2 - am^2.$

$\frac{m}{\sqrt{3}}$

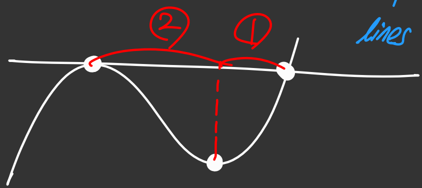
$= a(3x^2 - m^2).$

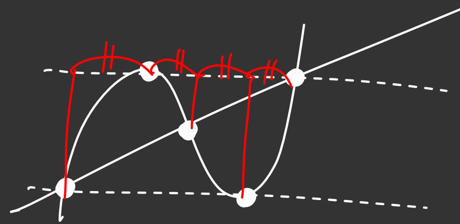
$-\frac{m}{\sqrt{3}}$

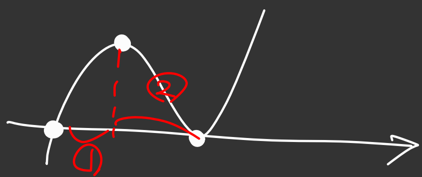
$= a(\sqrt{3}x - m)(\sqrt{3}x + m).$

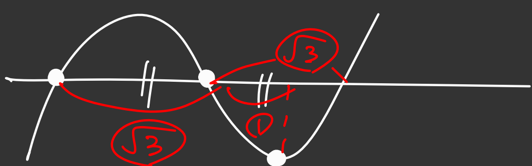
Zooming out. Visuals & ⊛

* the key is that it's a cubic & "A Horizontal Line" that is interacting! Break free from think the horizontal lines are all x-axis lines.







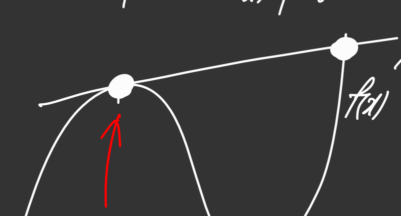


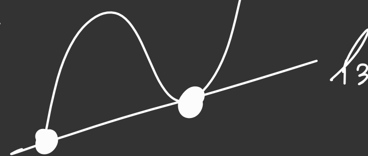
Vieta's formulae Epilogue:

let $l_\star = m_\star x + n_\star$ ($m_\star, n_\star \in \mathbb{R}$) and $f(x) = ax^3+bx^2+cx+d$.

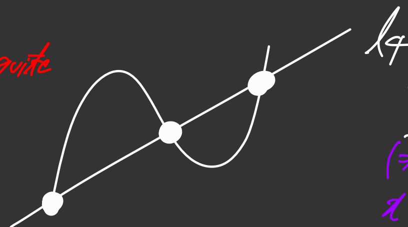
$f(x) - l_\star = ax^3+bx^2+(c-m_\star)x+(d-n_\star)=0$. → by vieta's formulae (sum of all roots)

$= -\frac{b}{a}$





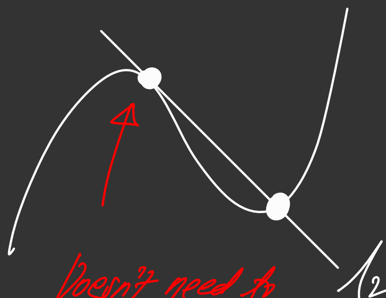
☆ What Matters: $-\frac{b}{a}$: coefficient of x^3, x^2.



X What Variable don't matter: c, d, m, n

(⇒ Coefficients of x and constant for both line and $f(x)$)

* Caveat: Repeat roots must quite literally be repeated.



Doesn't need to be local maximum/minimum!

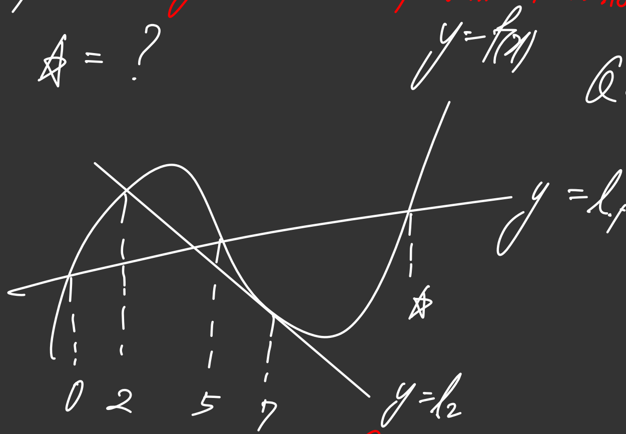
* The only caveat: Cubic $f(x)$ and line $l_\star$ must meet at at least 2 points in the graph.

Espresso • As long as the coefficients of x^3 and x^2 are constant, then the sum of all the roots are constant:

• all these graphs have the same sum being maintained.

Example 1. ☆ Using conservation of sum of roots for Cubic.

☆ = ?



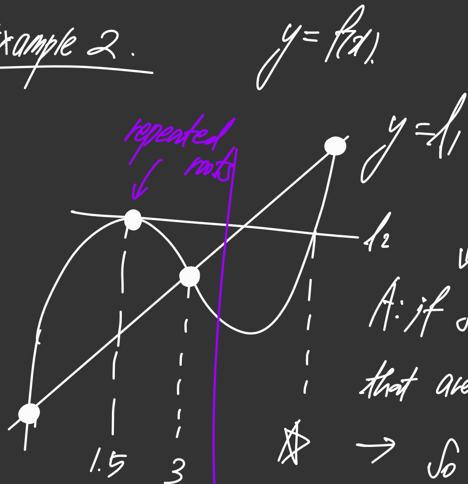
Q: $y=l_2$ is tangent to $y=f(x)$ at $x=7$. find ☆.

Count repeated roots twice.

$0+5+☆ = \underline{2+7+7}$

$\therefore ☆ = 11$

Example 2.



Q: if $y=l_1$ passes through inflection point, find ☆.

A: if so, then l_1 and $f(x)$ meet at 3 points such that are equidistant. (so are arithmetic sequences)

→ So $y=f(x)$ and $y=l_1$ meet at $3-d$, 3, $3+d$

inflection point.

Hence sum of 3 roots are $3-d+3+3+d$
$=9$.

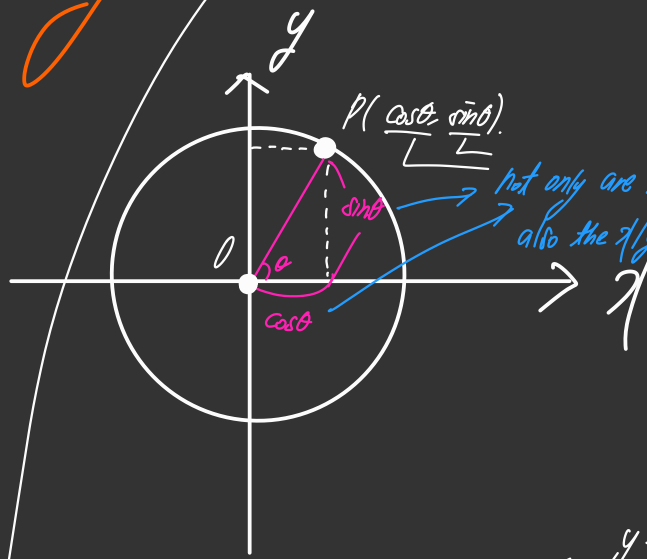
→ $9 = \underline{2\times(1.5)} + ☆$

$\therefore ☆ = 6$.

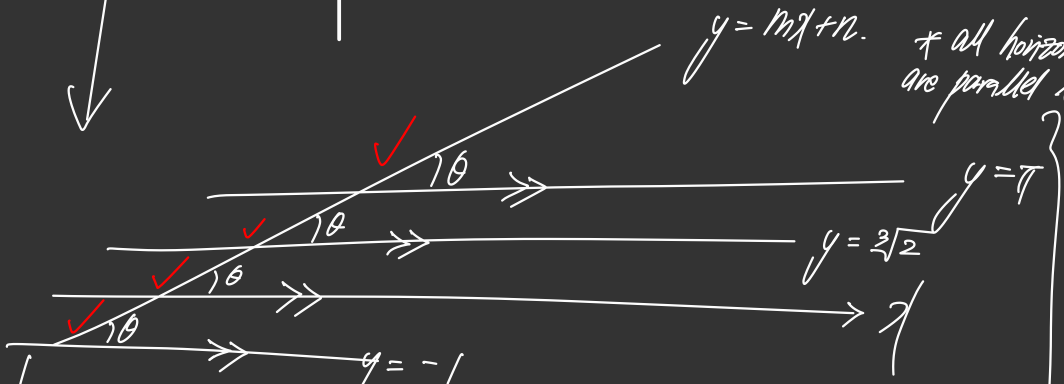
Trigonometry.

Trigonometry.

☆ Gradient of $\overline{OP}$ is going to be $(\tan\theta) \times x$.



not only are they lengths (scalar values) but are also the x/y coordinates itself (vectors with signs).



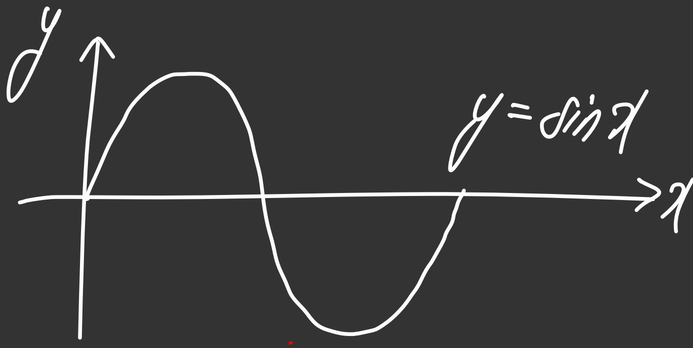
* all horizontal lines are parallel to each other.

all angles formed btw horizontal lines & $y = mx + n$ are equal to θ.

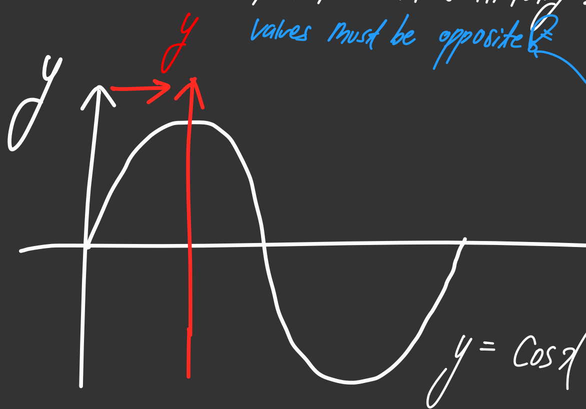
hence, m (coefficient of x at l) $= \tan\theta$. ($\because$ Corresponding Angles)

Generalisation : for every line l, let the angle it creates with the positive x axis $= \theta$

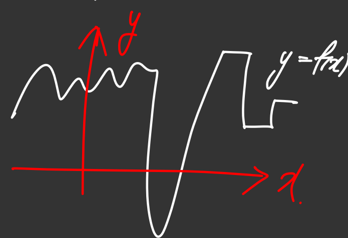
$\rightarrow$ Gradient of $l = \tan\theta$.



- TMUA9 • suggests a new frame. for graphs where it is hard to translate the function,, Don't! → Instead, translate the y-axis. (the coordinates aren't fixed break free from that scaffolding that x & y axis must be fixed. But the values must be opposite!



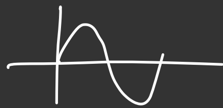
e.g) Imagine



if $f(x) \xrightarrow[y:-3]{x:+2}$ then just do
push x-axis to the left by 2.
push y-axis up by 3.

Analogy: $y = \cos x$ and $y = \sin x$ are just married. They are the perfect fit ($\sin^2 x + \cos^2 x = 1$), and also just by sliding $\sin x$ to the left by $\frac{1}{2}\pi$ gives $y = \cos x$, and just by pushing $\cos x$ to the right gives $y = \sin x$.

$y = \sin x$



- Point symmetric about the origin

⊕ Point symmetric about $(n\pi, 0)$

- Line symmetric about $x = \frac{2k+1}{2}\pi$ $(k \in \mathbb{Z})$

$y = \cos x$

- line symmetric about the y-axis.

⊕ line symmetric about $x = n\pi$

- Point symmetric about $\left(\frac{2k+1}{2}\pi, 0\right)$ $k \in \mathbb{Z}$

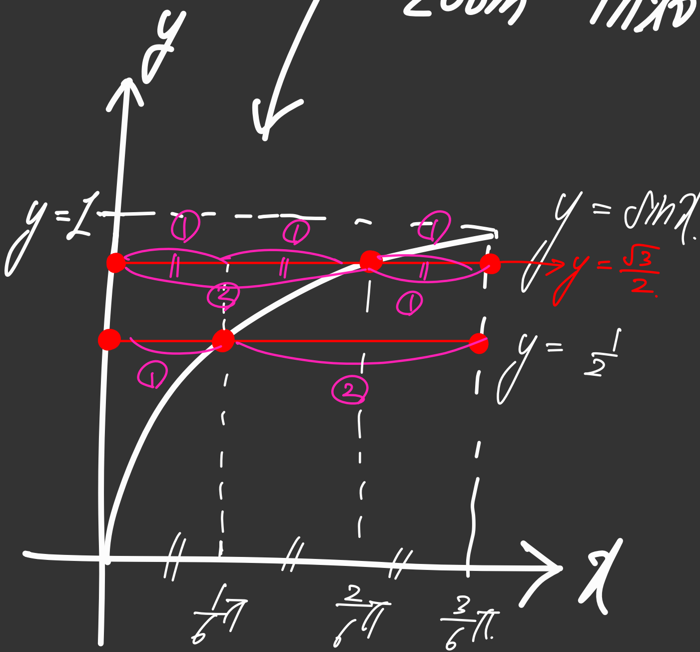
- $\sin(90^\circ - \theta) = \cos\theta$

- $\sin\theta = \cos(90^\circ - \theta)$

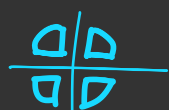
* both are line symmetric & point symmetric !

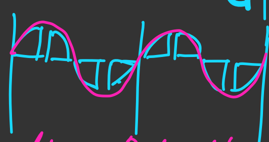
* both consist of 4 quarters of  (half a hemisphere shape).

Zoom into one block.



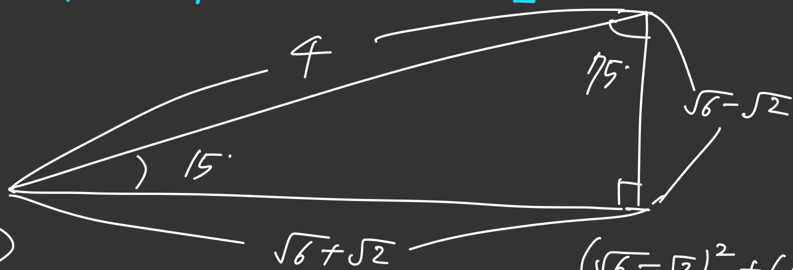
Both sine & cosine consist of this one block. So both sin x / cos x must be thought of as a rhythm of 4 blocks.

Visualisation: Think of a unit circle.  → 4 quarters of a circle.

→ And Rearrange these 4 circles.  cosine shape.

To be precise, it's not the same thing. But it's a good way for intuitive visualisation.

↗. 「TMUA9. Noir Premiere⊥ for Proof.



Pythagoras.

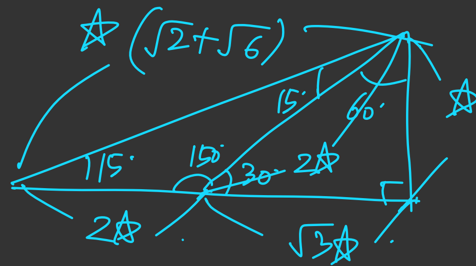
$$(\sqrt{6}-\sqrt{2})^2+(\sqrt{6}+\sqrt{2})^2$$
$$= 8-\sqrt{12}+8+\sqrt{12} = 16 = 4^2 \checkmark$$

15°

$$\tan 15 = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{1}{4}(\sqrt{6}-\sqrt{2})^2$$

$$\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4} = \cos(90^\circ-15^\circ) = \cos 75^\circ$$

$$\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} = \sin(90^\circ-15^\circ) = \sin 75^\circ$$



75°

$$\tan 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{1}{4}(\sqrt{6}+\sqrt{2})^2$$

$$\sqrt{6}-\sqrt{2}$$

$$\sin 75 = \frac{\sqrt{6}+\sqrt{2}}{4} = \cos(90^\circ-75^\circ) = \cos 15^\circ$$

$$\cos 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{4} = \sin(90^\circ-75^\circ) = \sin 15^\circ$$

Logic.

If A, then B

B, if A

A is sufficient for B

B is necessary for A

A is a subset of B

B is inclusive of A

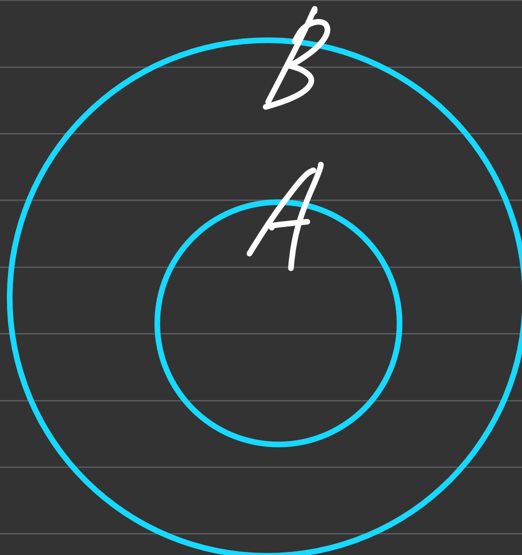
If A is true, then B is true

B is true if A

simply add (is true)

They're all the same.

$$A \to B.$$



9. TMUA9. Preaches a strong anti-thesis against thinking of logic in common sense. Have an algorithmic approach: Which structure amongst the above, is being presented? → Simply plug in values.

Axiomatic Truth(S).

Action steps for memorise the truth statements.

① Visualise. Draw them out (by hand) and see firsthand how each hold true.

② Memorise the statements, and also the following 3 must be known together!
* Conditions that the proposition holds true.
* Negate statement (yes, yourself), and know it is false.
* Contrapositive Statement (again yourself), and know it is true.

③ Prove why such are true in mathematical terms and the graphical perspective, NEVER only common sense.

e.g). let $f(x)$ is increasing.

(X) It ~~kinda~~ goes up.

What does go even mean?

What do you mean up?

(O_1) Mathematical Terms

if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

(O_2) Graphical Terms

if $\underbrace{x_2 - x_1 > 0}_{\equiv (\Delta x > 0)}$, then $\underbrace{f(x_2) - f(x_1) > 0}_{\equiv (\Delta y > 0)}$.

for a positive Δx from a point, Δy is a positive value.

True Statements. (Contrapositives on the next side)

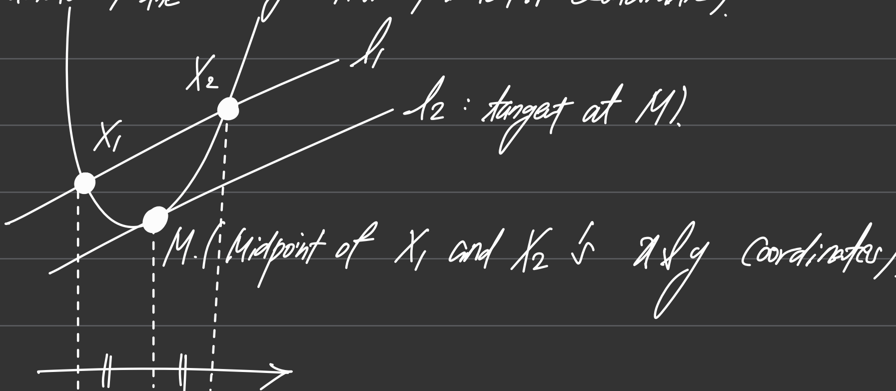
{Quadratics}.

- Every quadratic is line symmetric about vertical line that passes through the vertex.

- Every quadratic has one vertex.
- Every quadratic has no inflection point. (≡ Either entirely concave up or entirely concave down for the given domain).
- At the midpoint of the two points of intersections between a line and a quadratic, the tangent is parallel to the line.

x-Coordinate of the

$y = f(x) =$ Parabola (Quadratic).



- End behaviour of all cubic functions are always opposite signs.
  ↳ (at $x \to \pm\infty$, $f(x)$'s behaviour is called end behaviour).

{Cubics}.

- Every Cubic function is point symmetric about the inflection point.
- The shape of a cubic function is purely determined by the gradient of the tangent at the inflection point.
- For a line that passes through the inflection point of a Cubic function, it can only have an odd number of intersection with the cubic (1/3)

{ Polynomials }.

- If the product of the power of each terms are odd, then that polynomial is point symmetric about the origin.

e.g) let $f(x) = -x^2 + 7x$.

{Attitude}

- Elon Musk's 5-step cognitive algorithm for problem solving & Annotations.

① Make requirements less dumb : Osmosis (influx) of axiomatic truths & Dilution (outflow) of wrong statements.

② Delete the part or process. :

⇒ (1) lay out your entire compartmentalised thought process
(2) find ways to first try to delete entire process.
(if it can't hold, then we need to compartmentalise your thought process more).

③ Simplify & Optimise Design.
(maximise output generation thus optimisation, relative to focusing on streamlining input supply chain).
⇒ Just get to the answer, somehow someway.

④ Accelerate Cycle time ; cut $\frac{dx}{dt}$ (time between iteration of feedback loops).
⇒ regularly check pride & ego inflation.

⑤ Automate.
⇒ Generalisation comes last. Students either skip the process of generalisation entirely, or just start off with

- Think in absolutes and extremes first, then contract. Don't think the opposite way. (Don't think locally, anticipate global first to reverse engineer into the local)

e.g) I'm sketching a new function of f(x). But I can't seem to estimate how it is going to be...) ① You're thinking too locally.

→ So think in absolutes / extremes: send $x \to \infty$, what does f(x) do here?
$x \to -\infty$ " ?

- Dive head first into the impossibles (Especially for inequalities)

let's say domain is $7 < x$.

Start firsthand from $x = 7$. (the "impossibles"). and reverse engineer

- Don't flinch to question where it's not intuitively solvable.

The necessary and sufficient condition for a question to exist is an answer to exist. also for a problem and a solution. It's impossible that there is no output for any input, at least "no answer" must be generated. So knowing that truth that you can solve this question, and also ingesting everything from TMUA?, it is impossible that you can't solve something due to a knowledge gap.