TMUA9. Model Answer.

Cambridge Assessment
Admissions Testing

TEST OF MATHEMATICS
FOR UNIVERSITY ADMISSION

PAPER 1

SPECIMEN Time: 75 minutes

Additional Materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only points for correct answers, so you should attempt all 20 questions. Each question is worth one mark.

Any rough work should be done on this question paper. No extra paper is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

Calculators must **NOT** be used. There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page

This question paper consists of 12 printed pages and 4 blank pages

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1. The sum of the two values of x that satisfy the simultaneous equations

$x - 3y + 1 = 0$ and $3x^2 - 7xy = 5$ is

A -8.5

B -7.5

C -1.5

D 3.5

E 4.5

F 5

$3y = x+1$

$9x^2 - 7x(3y) = 15$

$9x^2 - 7x(x+1) = 15$

$2x^2 - 7x - 15 = 0.$

$2 \quad +3$

$1 \quad -5$

$-1.5 \quad 5$

$(2x+3)(x-5) = 0.$

2. The number of solutions in the interval $0 \le \theta \le 4\pi$ of the equation

$\sin^2\theta + 3\cos\theta = 3$ is

* Caveat: If you're going to substitute cos θ as c, then define domain. Manipulating as variable, you can lose sight that its in [-1,1].

$-1 \le c \le 1$

$1 - c^2 + 3c = 3.$

$c^2 - 3c + 2 = 0.$

$c = 1.$

A 0

B 1

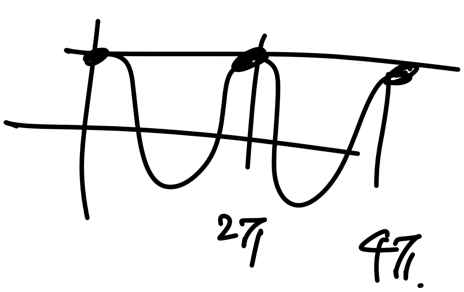
C 2

D 3

E 4

F 5

G 6



?

3. The perpendicular bisector of the line segment joining the points $(2, -6)$ and $(5, 4)$ cuts the x-axis at the point with x-coordinate

A $\quad \frac{1}{20}$

B $\quad \frac{1}{6}$

C $\quad \frac{1}{3}$

D $\quad \frac{19}{5}$

E $\quad \frac{41}{6}$

$$y = -\frac{2-5}{-6-4}(x-3.5)+(-1)$$

$$0 = -\frac{3}{10}(x-3.5)-1$$

$$\frac{3}{10}(x-3.5) = -1$$

$$x-3.5 = -\frac{10}{3}$$

$$x = \frac{7}{2}-\frac{10}{3} = \left(3+\frac{1}{2}\right)-\left(3+\frac{1}{3}\right).$$

?

4. The complete set of values of x for which $(x^2-1)(x-2) > 0$ is

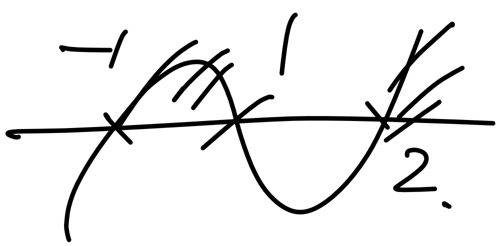
A $\quad x < -1,\ 1 < x < 2$

B $\quad x < -1,\ x > 2$

C $\quad -1 < x < 2$

D $\quad x < 1,\ x > 2$

E $\quad -1 < x < 1,\ x > 2$



Graph Not Necessary.

9

5. Given that $y = -\log_{10}(1-x)$ for $x < 1$, find x in terms of y.

$-y = \log_{10}(1-x)$

$10^{-y} = 1-x$

$x = 1 - 10^{-y}$

A $x = -\frac{1}{\log_{10}(1-y)}$

B $x = 1 + \log_{10} y$

C $x = 1 - \log_{10} y$

D $x = 1 - 10^{-y}$

E $x = 10^{-y} - 1$

F $x = 10^{1-y}$

* the premise is... think of graphs always. But here the intuition to discern when to sketch graphs & when not to. NOT BEING ABLE TO SKETCH A GRAPH MUST NOT BE AN EXCUSE.

9.

* factor theorem

6. It is given that $x + 2$ is a factor of $x^3 + 4cx^2 + x(c+1)^2 - 6$.

The sum of the possible values of c is

$f(x) = x^3 + 4cx^2 + x(c+1)^2 - 6$

$f(-2) = -8 + 16c - 2(c+1)^2 - 6$

$= 16c - 14 - 2c^2 - 4c - 2$

$= -2c^2 + 12c - 16$

$= -2(c^2 - 6c + 8)$

A -10

B -6

C 0

D 6

E 10

☆ When selecting balls type question, time matters! (Is it all taken out at once, or sequently?)

9.

7. A bag contains n red balls, n yellow balls, and n blue balls.

One ball is selected at random and not replaced.

A second ball is then selected at random and not replaced.

Each ball is equally likely to be chosen.

The probability that the two balls are **not** the same colour is

Complementary set

$$1-\left\{\frac{n}{3n}\times\frac{n-1}{3n-1}\times \underbrace{3}_{3 \text{ colours}}\right\}$$

A $\frac{n-1}{3n-1}$

B $\frac{2n-2}{3n-1}$

C $\frac{2n}{3n-1}$

$$= 1-\frac{n-1}{3n-1}$$

D $\frac{(n-1)^3}{27(3n-1)^3}$

E $\frac{3(n-1)}{3n-1}$

$$= \frac{3n-1-n+1}{3n-1}$$

F $\frac{n^3}{27(3n-1)^3}$

★ Same base / Same exponent → team them up!

9.

8. Given that $a^x b^{2x} c^{3x} = 2$, where a, b, and c are positive real numbers, then $x =$

A $\log_{10}\left(\frac{2}{a+2b+3c}\right)$

B $\frac{\log_{10} 2}{\log_{10}(a+2b+3c)}$

C $\frac{2}{\log_{10}(a+2b+3c)}$

D $\frac{2}{a+2b+3c}$

E $\log_{10}\left(\frac{2}{ab^2c^3}\right)$

F $\frac{\log_{10} 2}{\log_{10}(ab^2c^3)}$

G $\frac{2}{\log_{10}(ab^2c^3)}$

H $\frac{2}{ab^2c^3}$

$(ab^2c^3)^x = 2.$

$a^x = b$

$\log_a a^x (= x) = \log_a b$

$x = \log_{ab^2c^3} 2$

$\log_a b = \frac{\log_c b}{\log_c a}$

$= \frac{\log 2}{\log_{10} ab^2c^3}$

★ Vieta's formulae

let $ax^2+bx+c = a(x-\alpha)(x-\beta)$

$-a(\alpha+\beta) = \frac{b}{a}$, $a\alpha\beta = c$ → $\begin{cases} \alpha+\beta = -\frac{b}{a} \\ \alpha\beta = \frac{c}{a} \end{cases}$

9.

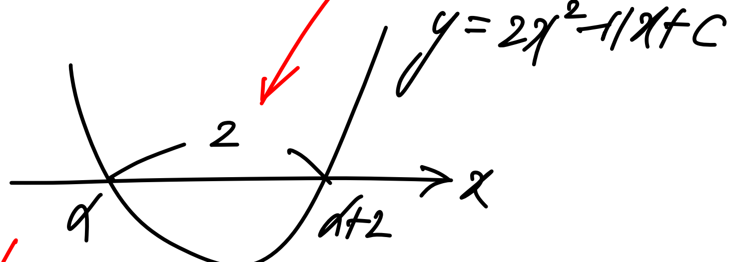
9. The roots of the equation $2x^2 - 11x + c = 0$ differ by 2. The value of c is

A $\frac{105}{8}$

B $\frac{113}{8}$

C $\frac{117}{8}$

D $\frac{119}{8}$



Vieta's formula → $\alpha + (\alpha+2) = \frac{11}{2}$

$2\alpha = \frac{11-4}{2} = \frac{7}{2}.$

$\alpha = \frac{7}{4}$

$\frac{c}{2} = \alpha(\alpha+2) = \frac{7}{4} \cdot \frac{15}{4} = \frac{105}{16}$, $c = \frac{105}{8}$

* Reflect $y=f(x)$ by $y=k$: Substitute $(2y-k)$ into y.
* Translate by * units in the positive x direction: $(x-k)$ into x.

9.

10. The curve $y = \cos x$ is reflected in the line $y = 1$ and the resulting curve is then translated by $\frac{\pi}{4}$ units in the positive x-direction. The equation of this new curve is

$$y = \cos x \xrightarrow[x:\, x-\frac{\pi}{4}]{y:\, 2-y} 2-y = \cos\left(x-\frac{\pi}{4}\right)$$

$$y = 2-\cos\left(x-\frac{\pi}{4}\right).$$

A $\quad y = 2 + \cos\left(x + \frac{\pi}{4}\right)$

B $\quad y = 2 + \cos\left(x - \frac{\pi}{4}\right)$

C $\quad y = 2 - \cos\left(x + \frac{\pi}{4}\right)$

D $\quad y = 2 - \cos\left(x - \frac{\pi}{4}\right)$

* When substituting by introducing new variable, MUST MUST DEFINE NEW DOMAIN.

9.

11. The sum of the roots of the equation $2^{2x} - 8 \times 2^x + 15 = 0$ is

let $2^x = \star$. $(\star > 0)$

$\star^2 - 8\star + 15 = 0$

$(\star - 3)(\star - 5) = 0.$

$\star = 3, 5$

$\therefore 2^x = 3, 5$

$x = \log_2 3, \log_2 5$

$\Sigma x \rightarrow \log_2(3\times 5) = \log_2 15 = \dfrac{\log_{10} 15}{\log_{10} 2}$

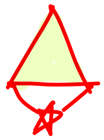
A $\quad 3$

B $\quad 8$

C $\quad 2\log_{10} 2$

D $\quad \log_{10}\left(\frac{15}{4}\right)$

E $\quad \dfrac{\log_{10} 15}{\log_{10} 2}$



· area of equilateral $= \frac{\sqrt{3}}{4} a^2$.

9.

12. The cross-section of a triangular prism is an equilateral triangle with side $2x$ cm. The length of the prism is d cm.

Let the total surface area of the prism be T cm². Given that the volume of the prism is T cm³, which one of the following is an expression for d in terms of x?

$\text{Area} = \frac{\sqrt{3}}{4}(2x)^2$

$= \sqrt{3}x^2$

total surface area　　total Volume.

$T = (2x)d \times 3 + 2 \times \sqrt{3}x^2 = \sqrt{3}x^2 \times d$

$\times \frac{1}{x}$

$6d + 2\sqrt{3}x = \sqrt{3}xd.$

A $\quad \frac{x}{2x-3}$

B $\quad \frac{3x}{3x-2\sqrt{3}}$

C $\quad \frac{2x}{x-4\sqrt{3}}$

D $\quad \frac{2x}{x-2\sqrt{3}}$ $\qquad 2\sqrt{3}x = (\sqrt{3}x - 6)d$

E $\quad \frac{2x}{x-\sqrt{3}}$

* this is what I mean when you should recognise cues that the examiner gives you.

split into

$f(x) = 0 \longrightarrow g(x) - h(x) = 0.$

I recognised the cue that $x^4 - 4x^3 + 4x^2$ as a function that is line symmetric

9.

13. How many real roots does the equation $x^4 - 4x^3 + 4x^2 - 10 = 0$ have?

$f(x) = g(x) - h(x)$, where

$g(x) = x^4 - 4x^3 + 4x^2$

$= x^2(x^2 - 4x + 4)$

$= x^2(x-2)^2.$

A $\quad 0$

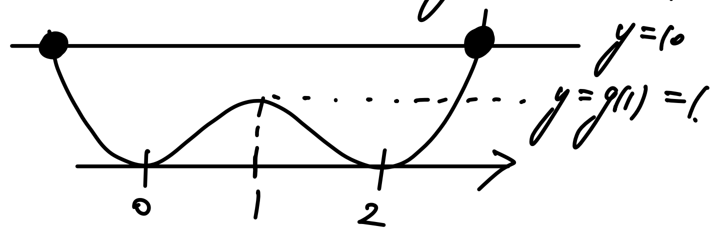
B $\quad 1$

C $\quad 2$

D $\quad 3$

E $\quad 4$

$g(x) = x^2(x-2)^2.$



[Turn over

☆ A against B. A → B. y against a.

9.

14. a, b, x, and y are real and positive.

a and b are constants.

x and y are related.

A graph of $\log y$ against $\log x$ is drawn.

For which one of the following relationships will this graph be a straight line?

A $\quad y^b = a^x$

B $\quad y = ab^x$

C $\quad y^2 = a + x^b$

D $\quad y = ax^b$

E $\quad y^x = a^b$

$\hookrightarrow \log y = m\log x + n \quad m\in\mathbb{R}, n\in\mathbb{R}.$

$= m\log x + n\log_{10} 10$

$= \log x^m + \log 10^n$

$\log y = \log(\underbrace{10^n}_{\text{Constant}} \times x^m)$

$y = 10^n \times x^m = ☆ \times x^m \cong a\cdot x^b$

☆, $m\in\mathbb{R}$

✓ Typical Question where the habit of leaning towards graph shines☆

☆ Pure Graph Play.

$f(x) = ax^2+\ldots$ } area of quadratic $= \frac{|a|}{6}☆^3$

9.

15. The smallest possible value of $\int_0^1 (x-a)^2\,dx$ as a varies is

] Inter PDF of written solutions

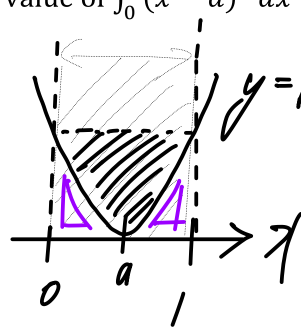
A $\quad \frac{1}{12}$

B $\quad \frac{1}{3}$

C $\quad \frac{1}{2}$

D $\quad \frac{7}{12}$

E $\quad 2$



$y=(x-a)^2$, so minimum at $a=\frac{1}{2}$.

$\rightarrow f(0) = \frac{1}{4}$.

$\frac{1}{4}\times 1 - \frac{1}{6}\times 1^3 = \frac{1}{12}$.

* think of it as rectangle with base length of 1 moving in relation to $y=(x-a)^2$. Obviously

*$ 2 & 5 power play show up all the time.
Habituate prime factorisation.

16. Given that c and d are non-zero integers, the expression $\dfrac{10^{c-2d} \times 20^{2c+d}}{8^{c} \times 125^{c+d}}$ is an integer if ______.

A $c < 0$

B $d < 0$

C $c < 0$ and $d < 0$

D $c < 0$ and $d > 0$

E $c > 0$ and $d < 0$

F $c > 0$ and $d > 0$

G $d > 0$

H $c > 0$

$$\frac{(2\times5)^{c-2d} \times (2^2\times5)^{2c+d}}{(2^3)^c \times (5^3)^{c+d}}$$

$$= 2^{(c-2d)+2(2c+d)-(3c)} \times 5^{(c-2d)+(2c+d)-3(c+d)}$$

$$= 2^{2c} \times 5^{-4d} \rightarrow \begin{cases} c>0 \\ d<0 \end{cases}$$

9

☆☆☆

*a little logic play here. in #16, it's given that d is integer. 5^{-4d} is an integer if

BUT d being a negative integer, is a SUFFICIENT but NOT NECESSARY condition for 5^{-4d} to be an integer.

why? try $-4d = \log_5 2$

$5^{\log_5 2} = 2^{\log_5 5} = 2.$

$a^{\log_b c} = c^{\log_b a}$

Just in case you did not know!

17. For what values of the non-zero real number a does the quadratic equation $ax^2 + (a-2)x = 2$ have real distinct roots?

9

A All values of a

B $a = -2$

C $a > -2$

D $a \neq -2$

E No values of a

$ax^2 + ax = 2x + 2.$

$ax(x+1) = 2(x+1)$

$(ax-2)(x+1) = 0$

↓ for this to not be $a(x+1)(x+1)=0,$

$ax-2 \neq ax+a$

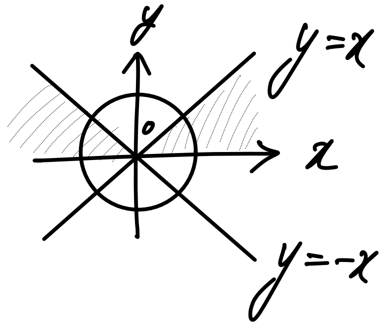
$\therefore a \neq -2$

$\log_x c =$

9

18. The angle x is measured in radians and is such that $0 \le x \le \pi$.

The total length of any intervals for which $-1 \le \tan x \le 1$ and $\sin 2x \ge 0.5$ is



at $x \in \{x \mid [0, \frac{1}{4}\pi] \cup [\frac{3}{4}\pi, \pi]\}$.

A $\frac{\pi}{12}$

B $\frac{\pi}{6}$

C $\frac{\pi}{4}$

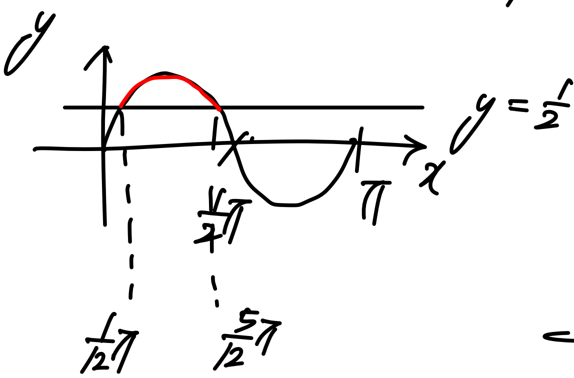
D $\frac{\pi}{3}$

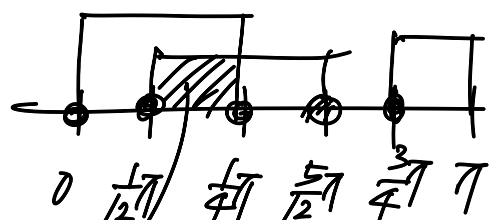
E $\frac{5\pi}{12}$

F $\frac{\pi}{2}$

G $\frac{5\pi}{6}$

$\sin(2x) \ge \sin(\frac{1}{6}\pi)$: at $x \in [\frac{1}{12}\pi, \frac{5}{12}\pi]$.





※ for arithmetic sequence a_n: $a_{n-k}, a_n, a_{n+k} \to 2a_n = a_{n-k} + a_{n+k}$

length of interval: $\frac{1}{4}\pi - \frac{1}{12}\pi = \frac{1}{6}\pi$.

19. A geometric series has first term 4 and common ratio r, where $0 < r < 1$. (so $r \neq 1$)

The first, second, and fourth terms of this geometric series form three successive terms of an arithmetic series.

The sum to infinity of the geometric series is

$$4 \xrightarrow{+☆} 4r \xrightarrow{+☆} 4r^3$$

$r^2 + r - 1 = 0$

$r = \frac{-1+\sqrt{5}}{2}$ ($\because 0<r<1$)

$\frac{4}{1-r} = \frac{4}{1-\frac{-1+\sqrt{5}}{2}} = 2(3+\sqrt{5})$

A $\frac{1}{2}(\sqrt{5}-1)$

B $2(3-\sqrt{5})$

C $2(1+\sqrt{5})$

D $2(3+\sqrt{5})$

sol 1)

$4r - 4 = 4r^3 - 4r$

$4(r-1) = 4r(r^2-1)$

$= 4r(r-1)(r+1)$

$r \neq 1$

$1 = r(r+1)$

sol 2) $2(4r) = 4 + 4r^3$

$2r = 1 + r^3$.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & -1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array}$$

$(r-1)(r^2+r-1)=0$.

☆ Mold. Morph, and Bend question into your favour.

9

20. The coefficient of x^2 in the expansion of $(4 - x^2)[(1 + 2x + 3x^2)^6 - (1 + 4x^3)^5]$ is

A 28

B 72

C 78

D 192

E 240

F 310

G 312

two avenues for term x^2 to show up

Current form: $(4-x^2) \times A(x)$.

① $4 \times (\star x^2$ ($\star$ = real constant) term for $A(x)$).

② $(-x^2) \times$ (Constant term from $A(x)$)

Narrow down on aim:

Aim is to find two things: ((1) coefficient of x^2, (2) constant) at $A(x)$'s expansion.

END OF TEST

① x^2 coeff:

$A(x) = (1+2x+3x^2)^6 - (1+4x^3)^5$

all exponents of this one's expanded form are divisible by 3. so they have nothing to do with x^2 exponent.

$(3x^2)^1(2x)^0(1)^5 \times {}_6C_1 \times {}_5C_5$

$+ (3x^2)^0(2x)^2(1)^4 \times {}_6C_2 \times {}_4C_4$

$= 3 \times 6x^2 + 4 \times 15x^2 = 78x^2$.

② Constant at $A(x)$.

$(1)^6(3x^2)^0(2x)^0 \times {}_6C_6 - 1 = 0$.

$A(x) = 0 + \star x + 78x^2 + \cdots$

(in ascending order)

$(4-x^2)(0 + ?x + 78x^2 \cdots)$

$0 + (4 \times 78x^2 - 0x^2)$

$?x +$

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