Step 2: 3-Step Analysis

Cambridge Assessment Admissions Testing

TEST OF MATHEMATICS FOR UNIVERSITY ADMISSION **D513/11**

PAPER 1

Wednesday 8 November 2017 Time: 75 minutes

Additional Materials: Answer sheet

INSTRUCTIONS TO CANDIDATES

Please read these instructions carefully, but do not open the question paper until you are told that you may do so.

A separate answer sheet is provided for this paper. Please check you have one.
You also require a soft pencil and an eraser.

This paper is the first of two papers.

There are 20 questions on this paper. For each question, choose the one answer you consider correct and record your choice on the separate answer sheet. If you make a mistake, erase thoroughly and try again.

There are no penalties for incorrect responses, only marks for correct answers, so you should attempt all 20 questions. Each question is worth one mark.

Any rough work should be done on this question paper. No extra paper is allowed.

Please complete the answer sheet with your candidate number, centre number, date of birth, and full name.

Calculators and dictionaries must **NOT** be used.
There is no formulae booklet for this test.

Please wait to be told you may begin before turning this page.

This question paper consists of 21 printed pages and 3 blank pages.

PV2

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$(x)\ y = x^3 - \int \frac{2-3x}{x^3}\,dx$. Just simply simplify first.

1 Given that

$$\frac{dy}{dx} = 3x^2 - \frac{2-3x}{x^3}, \quad x \neq 0$$

and $y = 5$ when $x = 1$, find y in terms of x.

A $y = \frac{1}{3}x^3 + x^{-2} - 3x^{-1} + 6\frac{2}{3}$

B $y = x^3 + \frac{1}{2}x^{-2} - 3x^{-1} + 6\frac{1}{2}$

C $y = x^3 + x^{-2} - 3x^{-1} + 6$

D $y = x^3 + x^{-2} - x^{-1} + 4$

E $y = x^3 + 2x^{-2} - x^{-1} + 3$

F $y = 3x^3 + x^{-2} - x^{-1} + 2$

$$\frac{dy}{dx} = 3x^2 - (2-3x)x^{-3}$$

$$= 3x^2 - 2x^{-3} + 3x^{-2}$$

Integrating both sides.

$$y = x^3 - 2\frac{x^{-2}}{-2} - 3x^{-1} + C$$

$$= x^3 + x^{-2} - 3x^{-1} + C$$

* weirdly enough you could have skipped substitution entirely, and just ticked C and moved on to next question. Poorly constructed question.

at $x=1$, $y=5$

$5 = 1 + 1 - 3 + C \quad C = 6$

$f(1)$ asked $\neq$ find $f(x)$ and sub in $x=1$.
$f''(1)$ asked $\neq$ find $f(x)$ and sub in $x=1$.

2 The function f is given by

$$f(x) = \left(\frac{2}{x} - \frac{1}{2x^2}\right)^2 \quad (x \neq 0)$$

What is the value of $f''(1)$?

A -3

B -1

C 5

D 17

E 29

F 80

Nominal Solution) Just use chain rule.
$\to f'(x) = \sim$.
again chain rule.
Sub $x=1$ into $f''(x)$

Sol (9.))
We are trying to find a function value of $f''(x)$ at $x=1$. This does not mean we must find $f''(x)$ in an explicit function form.

Sol 2) $f(x) = \left(\frac{4x-1}{2x^2}\right)^2$

We solved it via pure $f(1), f'(1), f''(1)$ value comparison!

$\times (2x^2)^2$

$4x^4 f(x) = (4x-1)^2 \xrightarrow[x=1]{\text{Substitute}} 4f(1) = 9$. * Let go of the habit of finding $f(x) = \sim$.

Differentiate

$16x^3 f(x) + 4x^4 f'(x) = 2(4x-1) \times 4 = 8(4x-1) \xrightarrow[x=1]{\text{Substitute}} \underbrace{16 f(1)}_{36} + \underbrace{4f'(1)}_{-12} = 24$

Differentiate again!

$\left(48x^2 f(x) + 16x^3 f'(x)\right) + \left(16x^3 f'(x) + 4x^4 f''(x)\right) = 32$

$f'(1) = -3$

Substitute $x=1$: $48f(1) + 16f'(1) + 16f'(1) + 4f''(1) = 32$.

$f''(1) = 5$

3 A line l has equation $y = 6 - 2x$

A second line is perpendicular to l and passes through the point $(-6, 0)$.

Find the area of the region enclosed by the two lines and the x-axis.

A $16\frac{1}{5}$

B 18

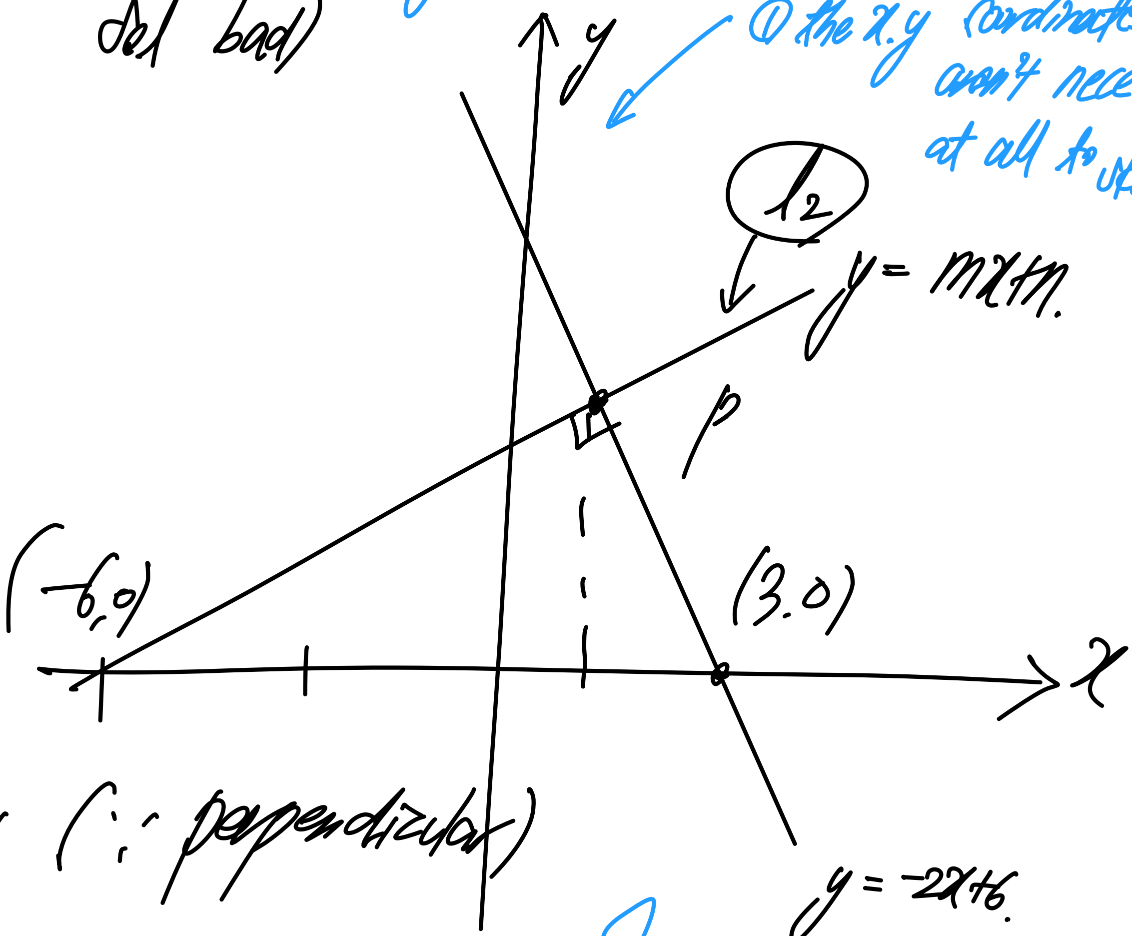
C $21\frac{3}{5}$

D 27

E $40\frac{1}{2}$

del bad)

Why this solution is bad:
① the x, y coordinates aren't necessary at all to sketch.



$-2m = -1$ ($\because$ perpendicular)

$m = \frac{1}{2}$

so l_2: $y - 0 = \frac{1}{2}(x + 6)$

Equating y, $-2x + 6 = \frac{1}{2}(x + 6)$

$-4x + 12 = x + 6$

$6 = 5x$

Hence $p\left(\frac{6}{5}, \frac{18}{5}\right)$.

② finding height isn't only done by y coordinate of p:

height of Δ

$= y_p = \frac{18}{5}$

$\underbrace{(3 - (-6))}_{\text{Base}} \times \underbrace{\frac{18}{5}}_{\text{height.}} \times \frac{1}{2}$

① $y = mx+n$
② $ax+by+c=0$
③ $y - f(t) = f'(t) \times (x-t)$
④ $\frac{x}{x_{int}} + \frac{y}{y_{int}} = 1.$ → Explicitly using Relationship with x-y axis.
⑤ $y - q = m(x-p)$

} Several types of linear function expressions & when to use which.

① Standard form., likewise for ②
③ when a line is a tangent at point $(t, f(t))$ to a function $y=f(x)$.
cf) when a line is normal at point $(t, f(t))$, that gives
$y = -\frac{1}{f'(t)}(x-t) + f(t)$

④ not taught in A-levels.

$\frac{x}{x_{int}} \boxed{+} \frac{y}{y_{int}} = \boxed{1}$

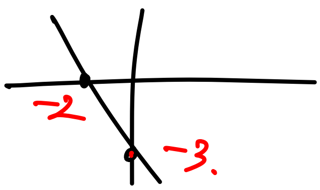
it must stay + & △, so sign must be positive.

s.t. x_{int}: x-intercept, y_{int}: y-intercept

e.g.①) $\frac{x}{1} + \frac{y}{1} = 1$
means (x-intercept, y-intercept) = (1,1)

① Plot the x & y intercepts first.
② Sketch line passing through these.

e.g.②) $\frac{x}{(-2)} + \frac{y}{(-3)} = 1$



A ☐ inter TMUA 2017 Paper 2. #2.

you do not need to specify perfect graph.

#kill this obsession.

3 A line l has equation $y = 6 - 2x$

A second line is perpendicular to l and passes through the point $(-6, 0)$.

★ Angle //

Find the area of the region enclosed by the two lines and the x-axis.

A $16\frac{1}{5}$

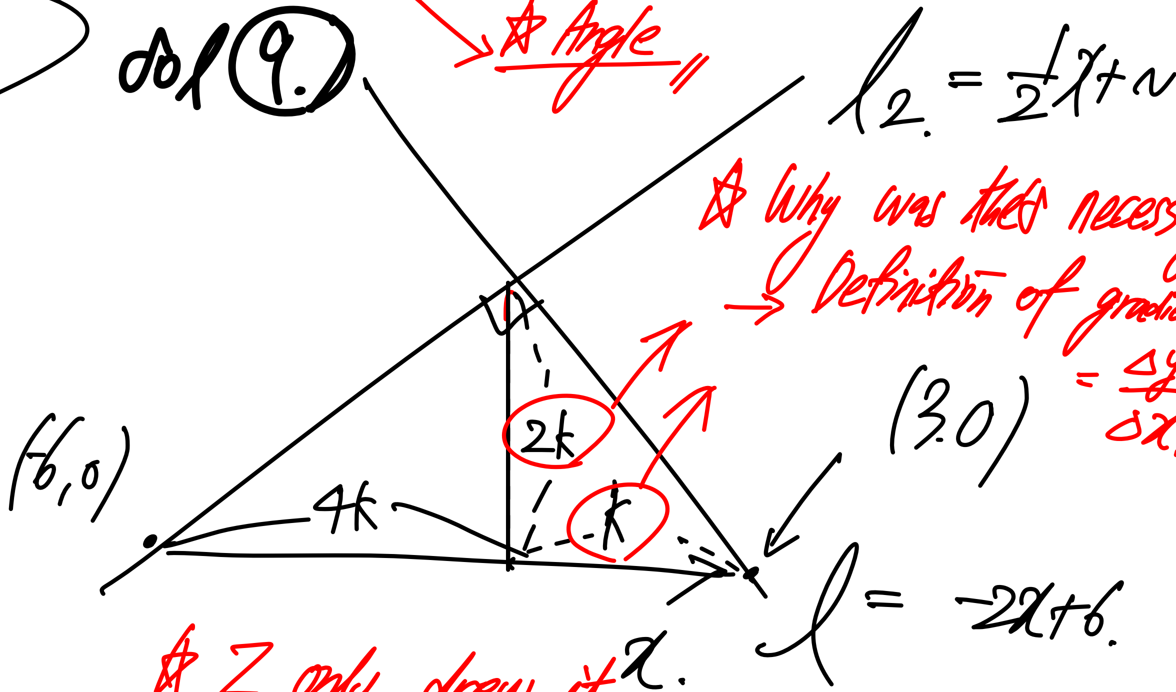
B 18

C $21\frac{3}{5}$

D 27

E $40\frac{1}{2}$

sol 9.



★ Why was that necessary → Definition of gradient $= \frac{\Delta y}{\Delta x}$.

★ I only drew it because (-6,0) is on the x-axis and the area of interest is on the x-axis.

$4k + k = |3-(-6)| = 5k = 9$

$\frac{5k \cdot 2k}{2} = 5k^2 = \frac{25k^2}{5} = \frac{81}{5} = 16\frac{1}{5}$

$\therefore$ Area(total) $= 16\frac{1}{5}$

4 When $(3x^2 + 8x - 3)$ is multiplied by $(px - 1)$ and the resulting product is divided by $(x + 1)$, the remainder is 24.

What is the value of p?

A -4

B 2

C 4

D $\frac{8}{7}$

E $\frac{11}{4}$

sol X)

$$\frac{(3x^2+8x-3)(px-1)}{(x+1)} = P(x) + \frac{24}{(x+1)}$$

(Not mathematically wrong but only adds useless chains of thought).

Jump straight into polynomial identity of Division statement ∮

$$\underbrace{P(x)}_{\text{Dividend polynomial}} = \underbrace{D(x)}_{\text{Divisor polynomial}}\underbrace{Q(x)}_{\text{Quotient polynomial}} + \underbrace{R(x)}_{}$$ → Remainder Polynomial.

Dividend polynomial

Quotient polynomial.

Divisor polynomial

- Selection of blocks of thought to minimize friction.

immediate Reaction Circuits

→ Division Check & factor theorem

4 When $(3x^2 + 8x - 3)$ is multiplied by $(px - 1)$ and the resulting product is divided by $(x + 1)$, the remainder is 24.

What is the value of p?

A -4

B 2

C 4

D $\frac{8}{7}$

E $\frac{11}{4}$

$$(3x^2+8x-3)(px-1) = (x+1)P(x) + 24$$

at $x=-1$

$$(3-8-3)(-p-1) = 24$$

$$-8(-p-1) = 24$$

$$8(p+1) = 24$$

$$p = 2.$$

* think in terms of sets.

① (Inequality 1) and ② (Inequality 2) ≡ ① $\cap$ ②

5 S is the complete set of values of x which satisfy **both** the inequalities

$$x^2 - 8x + 12 < 0 \text{ and } 2x + 1 > 9$$

) they explicitly said set so it's obvious to think in sets.

The set S can also be represented as a single inequality.

Which one of the following single inequalities represents the set S ?

** But even questions not given in sets, still thinking in terms of sets is ideal & simple.

A $(x^2 - 8x + 12)(2x + 1) < 0$

B $(x^2 - 8x + 12)(2x + 1) > 0$

C $x^2 - 10x + 24 < 0$

D $x^2 - 10x + 24 > 0$

E $x^2 - 6x + 8 < 0$

F $x^2 - 6x + 8 > 0$

G $x < 2$

H $x > 6$

①: $x^2 - 8x + 12 < 0$

$(x-2)(x-6) < 0$

$2 < x < 6$

②: $2x + 1 > 9$

$x > 4$

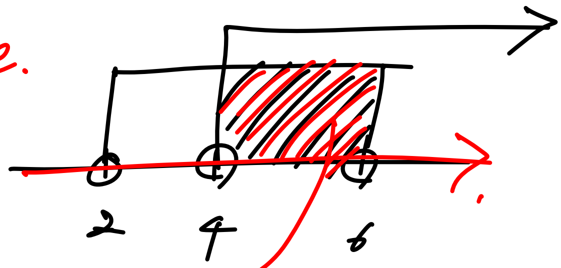
* ideal process here

① Plot all numbers on horizontal number line.

② {set 1} $\cap$ {set 2}

↳ find the AREAS OF INTERSECTION.

$\{2 < x < 6\} \cap \{x > 4\}$



$X = \{x \mid 4 < x < 6\}$

$(x-4)(x-6) < 0$

$x^2 - 10x + 24 < 0$.

When a function is articulated with terms that imply Graphical methods. pick up on it.

6 A tangent to the circle $x^2 + y^2 = 144$ passes through the point $(20, 0)$ and crosses the positive y-axis.

What is the value of y at the point where the tangent meets the y-axis?

A 12

B 15

C $\frac{49}{3}$

D 20

E $\frac{64}{3}$

F $\frac{80}{3}$

(Nominal Solution)

* Solution of not picking up on cues of graphs.

(in pure algebra).

let point of intersection be (x_t, y_t): $x_t^2 + y_t^2 = 144$

Gradient of tangent: $2x + 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{x}{y}$

$-\frac{x_t}{y_t}(x - x_t) + y_t = y$.

⋮

ideal Solution:

*let $\begin{cases} \{a_n\} = \text{Arithmetic Progression}: a_n, a_{n+1}, a_{n+2} \rightarrow 2a_{n+1} = a_n + a_{n+2} \\ \{b_n\} = \text{Geometric Progression}: b_n, b_{n+1}, b_{n+2} \rightarrow (b_{n+1})^2 = b_n b_{n+2} \end{cases}$

7 The first three terms of an arithmetic progression are p, q and p^2 respectively, where $p < 0$

The first three terms of a geometric progression are p, p^2 and q respectively.

Find the sum of the first 10 terms of the arithmetic progression.

A $\frac{23}{8}$

B $\frac{95}{8}$

C $\frac{115}{8}$

D $\frac{185}{8}$

(Nominal Solution)

$* \; p = a$

$q = a+d, \quad d = q - a = q - p$

$p^2 = a + 2d, \quad p^2 = p + 2(q-p) = 2q - p$ ①

$* \; p = b$

$p^2 = br, \quad r = \frac{p^2}{b} = p$

$q = br^2, \quad q = p \cdot p^2 = p^3$ ②

then solving ① & ② simultaneously.

using this

Model Answer)

$\begin{cases} 2q = p + p^2 \\ (p^2)^2 = p \times q \end{cases}$

$p^3 = q$

$2p^3 = p + p^2$

($p < 0$ thus $p \neq 0$)

$2p^2 = 1 + p$

$2p^2 - p - 1 = 0 \longrightarrow (2p+1)(p-1) = 0$

$\begin{matrix} 2 & 1 \\ 1 & -1 \end{matrix}$

$p = -\frac{1}{2} < 0$

$q = -\frac{1}{8}$

$a = p = -\frac{1}{2}$

$d = q - p = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$

$\frac{a_1 + a_{10}}{2} \times 10$

$= (2a + 9d) \times 5$

$= \left(-1 + \frac{27}{8}\right) \times 5$

$= \frac{95}{8}$

formulae.

let $\{a_n\}$ = arithmetic sequence, $\rightarrow$ $2a_{n+1} = a_n + a_{n+2}$

$\{b_n\}$ = geometric sequence, $\rightarrow$ $b_{n+1}^2 = b_n \times b_{n+2}$.

PROOF

$$a_n = a + (n-1)d$$

$$a_{n+1} = a + (n+1-1)d = a + nd$$

$$a_{n+2} = a + (n+2-1)d = a + (n+1)d.$$

$$2a_{n+1} = 2a + 2nd = 2a + (n+1+n-1)d$$
$$= (a + (n-1)d) + (a + (n+1)d)$$
$$= a_n + a_{n+2}.$$

✓

you MUST. MUST know the proof & meaning behind it first prior to formulae & mnemonics. Because knowing the meaning & the why is everything

Mnemonics (tip for memorisation)

✓Arithmetic Sequence: Double middle. one = sum of adjacent ones.

$2 \times a_{n+1}$ $\quad$ $a_n + a_{n+2}$.

✓Geometric Sequence: Squared middle one = product of adjacent ones.

a_{n+1}^2 $\quad$ $a_n \times a_{n+2}$.

Caveats: they must be. sequential.

#But❗ As long as they are constantly apart, it is also valid.

e.g). $\boxed{\text{let } \{a_n\} = \text{Arithmetic Sequence:}}$

Formulae:

$2a_2 = a_1 + a_3$: $a_1 \xrightarrow{+d} a_2 \xrightarrow{+d} a_3$

$2a_3 = a_2 + a_4$: $a_2 \xrightarrow{+d} a_3 \xrightarrow{+d} a_4$

$2a_4 = a_3 + a_5$: $a_3 \xrightarrow{+d} a_4 \xrightarrow{+d} a_5$

$2a_{n+1} = a_n + a_{n+2}$.

Proof

$a_{n+1} - a_n = a_{n+2} - a_{n+1}$
$= d$

mnemonic: Doubling middle one gives sum of adjacent ones.

#BUT❗ Beyond that

$2a_3 = a_1 + a_5$: $a_1 \xrightarrow{+2d} a_3 \xrightarrow{+2d} a_5$

$2a_5 = a_2 + a_8$: $a_2 \xrightarrow{+3d} a_5 \xrightarrow{+3d} a_8$

(proof)

$a_3 - a_1 = a_5 - a_3 = 2d$

$a_7 - a_4 = a_4 - a_1 = 3d$

Analogy: Grasshopper is jumping around stepping stones

Generalisation❗

$$a_n - a_{n-k} = a_{n+k} - a_n = kd.$$

$a_{n-k} \quad a_{n-k+1} \quad \cdots \quad a_{n-1} \quad a_n \quad a_{n+1} \quad \cdots \quad a_{n+k}$

$a_{n-k} \xrightarrow{+kd} a_n \xrightarrow{+kd} a_{n+k}$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8$

$a_1 \xrightarrow{+2d} a_3 \xrightarrow{+2d} a_5 \xrightarrow{+2d} a_7$

$a_2 \xrightarrow{+3d} a_5 \xrightarrow{+3d.} a_8$

Let's try Geometric Sequence!

$\boxed{\text{let } \{b_n\} = \text{Geometric Sequence.}}$

formulae:
$$b_{n+1}^2 = b_n b_{n+2}$$

$a_2^2 = a_1 a_3$: $a_1 \xrightarrow{\times r} a_2 \xrightarrow{\times r} a_3$

$a_3^2 = a_2 a_4$: $a_2 \xrightarrow{\times r} a_3 \xrightarrow{\times r} a_4$

$a_4^2 = a_3 a_5$: $a_3 \xrightarrow{\times r} a_4 \xrightarrow{\times r} a_5$

Proof
$$\left(\frac{b_{n+1}}{b_n} = \frac{b_{n+2}}{b_{n+1}} = r \right)$$

mnemonic: squaring middle one gives product of adjacent ones.

But ~~ff~~ we could do the same with geometric sequence.

$a_3^2 = a_1 a_5$: $a_1 \xrightarrow{\times r^2} a_3 \xrightarrow{\times r^2} a_5$

$a_4^2 = a_1 a_7$: $a_1 \xrightarrow{\times r^3} a_4 \xrightarrow{\times r^3} a_7$

Generalisation

$$\frac{a_n}{a_{n-k}} = \frac{a_{n+k}}{a_n} = r^k$$

$$a_{n-k} \quad a_{n-k+1} \quad \cdots \quad a_{n-1} \quad a_n \quad a_{n+1} \quad \cdots \quad a_{n+k-1} \quad a_{n+k}$$

$$a_{n-k} \xrightarrow{\times r^k} a_n \xrightarrow{\times r^k} a_{n+k}$$

Juxtaposition

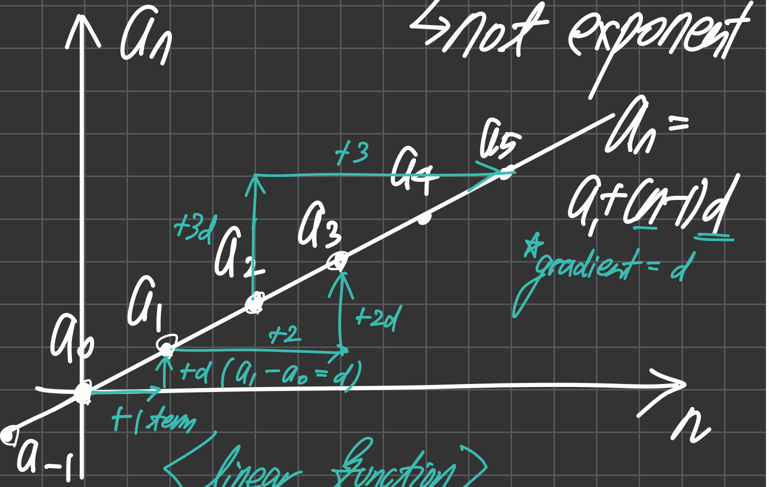
(I was gonna go navy for Oxford vs Cambridge but navy blends into black lol)

[Arithmetic Sequence]

Addition.

$\{a_n\} = a_1 \oplus \underline{(n-1)}d$

↳ not exponent



<linear function>

s.t gradient (= rate of change) = d.

You may / could add a_0, a_{-1} for more simple calculations.

$a_{n-k} \xrightarrow{+kd} a_n \xrightarrow{+kd} a_{n+k}$

$kd = a_n - a_{n-k} = a_{n+k} - a_n$ #Subtraction between terms are constant

↳ $2a_n = a_{n-k} + a_{n+k}$ ☆

e.g) $2a_2 = a_1 + a_3$ $(2)\times 2 = 1+3$

$2a_0 = a_{-1} + a_1$ $(0)\times 2 = -1+1$

$2a_4 = a_{-2} + a_{10}$ $(4)\times 2 = -2+10$

$n\times 2 = (n-k)+(n+k)$

$2n = 2n$ //

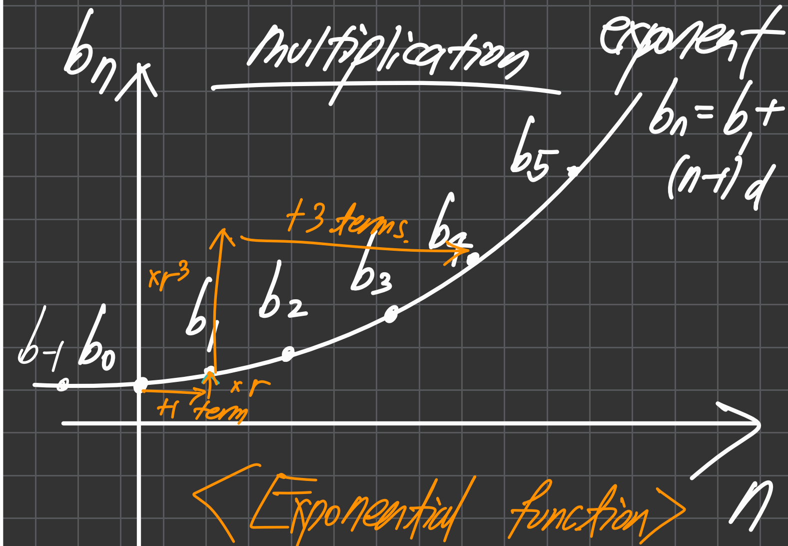
=

[Geometric Sequence]

$\{b_n\} = b_1 \otimes r^{n-1}$

multiplication

↳ ZS exponent



<Exponential function>

s.t base of exponential function is r.

likewise, it's more flexible for you to go off the grid and plot points beyond the first quadrant and adventure into 2nd Quadrant.

$b_{n-k} \xrightarrow{\times r^k} b_n \xrightarrow{\times r^k} b_{n+k}$

$r^k = \dfrac{b_n}{b_{n-k}} = \dfrac{b_{n+k}}{b_n}$ # Division between terms are constant.

$b_n^2 = b_{n-k} b_{n+k}$ ☆ $2\times n = (n-k)+(n+k)$

e.g) $b_2^2 = b_1 b_3$ $(2)\times 2 = 1+3$

$b_{-1}^2 = b_{-4} b_2$ $(-1)\times 2 = -4+2$

$b_0^2 = b_{-10} b_{10}$ $(0)\times 2 = -10+10$

for inequalities that are in product form Divide into cases.

e.g) $f(x)g(x) \geq 0$. i) $\{f(x) \geq 0\} \cap \{g(x) \geq 0\}$
ii) $\{f(x) \leq 0\} \cap \{g(x) \leq 0\}$) # As written, it is ideal to think in sets.

8 Find the complete set of values of x, with $0 \leq x \leq \pi$, for which

$$(1 - 2\sin x)\cos x \geq 0$$

A $0 \leq x \leq \frac{\pi}{6}$, $\frac{\pi}{2} \leq x \leq \frac{5\pi}{6}$

B $0 \leq x \leq \frac{\pi}{6}$, $\frac{5\pi}{6} \leq x \leq \pi$

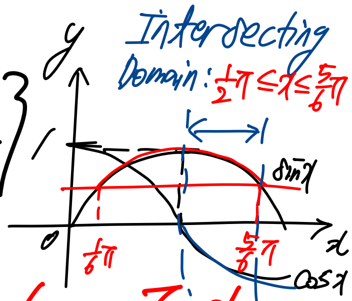
C $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$, $\frac{5\pi}{6} \leq x \leq \pi$

D $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

Sol(1) <Case Division>

(i) $\{1 - 2\sin x \geq 0\} \cap \{\cos x \leq 0\}$,

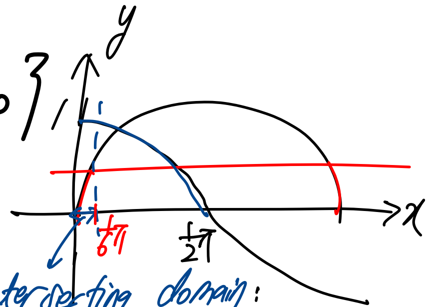
$\frac{1}{2} \geq \sin x$, $\cos x \leq 0$.



You could go graph or algebra here. I chose graph because I memorise key points. Laborious Algebra is still fine.

(ii) $\{1 - 2\sin x \leq 0\} \cap \{\cos x \geq 0\}$,

$\frac{1}{2} \leq \sin x \cap \cos x \geq 0$.



$0 \leq x \leq \frac{1}{6}\pi$

(i) ∪ (ii) gives A ▨

Sol2) let him cook.

$\cos x - 2\sin x \cos x \geq 0$.

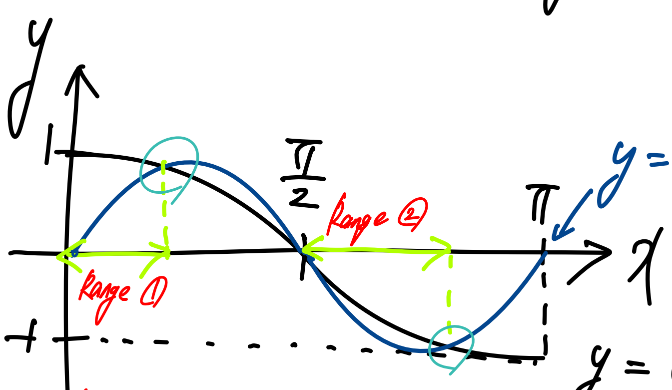
Also seen in #18.

$\cos x - \sin 2x \geq 0$.

$y = f(kx)$ is $y = f(x)$ squeezed by factor of $\frac{1}{k}$.

$\cos x \geq \sin 2x$ ← $y = \sin 2x$ is $y = \sin x$ stretched by a scale factor of $\frac{1}{2}$. (squeezed by half)

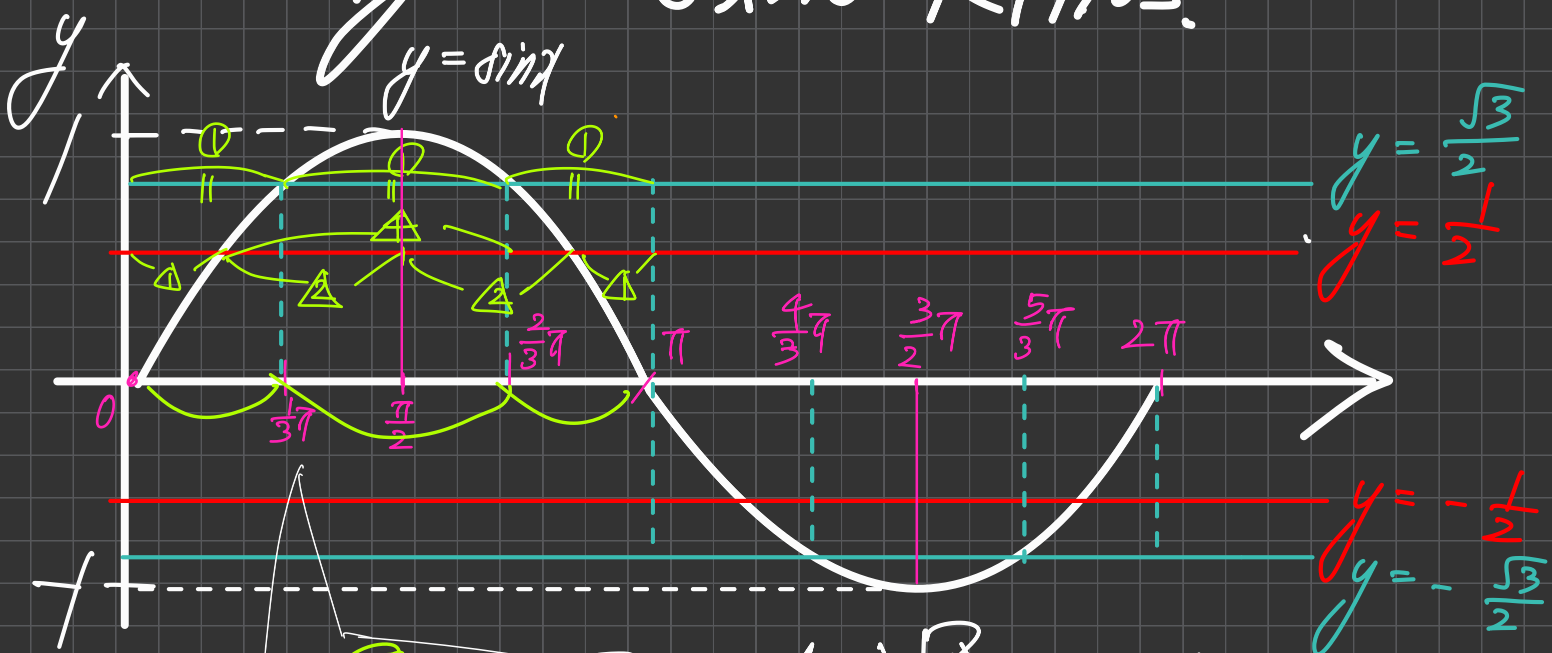
(* I only drew up to $0 \leq x \leq \pi$ as that was given domain in Question)



At {Range ①} ∪ {Range ②}, $y = \cos x$ is above $y = \sin 2x$

Tips: in pure logic, you can't derive x-coordinates solely off of graph, so a proper solution eventually goes back to algebra of solving $\cos x = \sin 2x$. But, again TMUA is MCQ. A is the most obvious solution (∵ $0 \leq$ ~).

Cool Trigonometric Ratios.

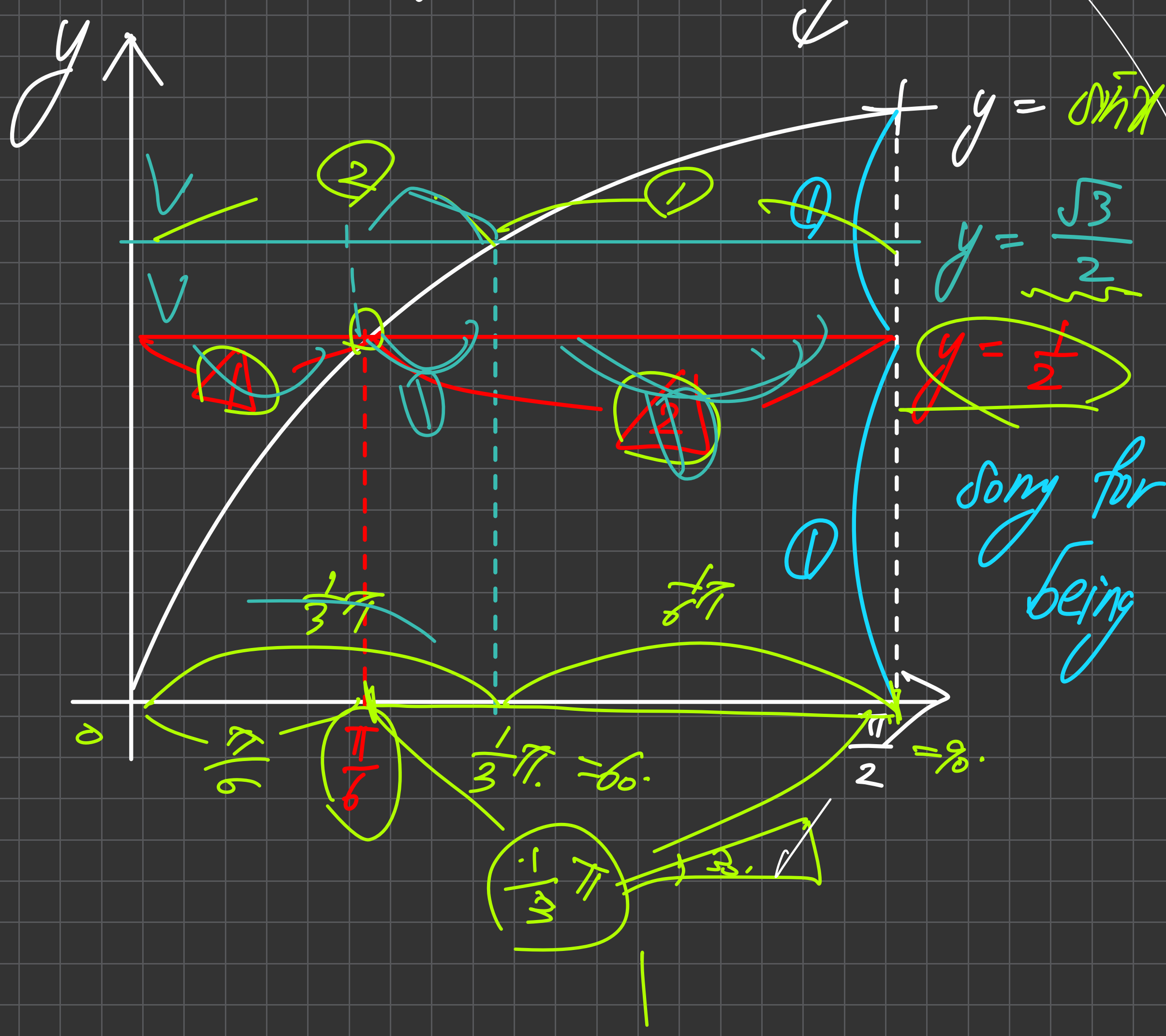


zoomed in!

$\int_0^{\frac{\pi}{2}} \sin x \, dx = [\cos x]_0^{\frac{\pi}{2}} = -(0-1) = 1$

this translated by $-\frac{\pi}{2}$ in $+x$ axis gives $y = \cos x$.

Both $y = \sin x$ & $y = \cos x$ consist of repeated "blocks"!



sorry for /// not being at scale.

* If you're solving this with algebra, you're having an affair at that point.

9 A circle has equation $x^2 + y^2 - 18x - 22y + 178 = 0$

A regular hexagon is drawn inside this circle so that the vertices of the hexagon touch the circle.

What is the area of the hexagon?

A 6

B $6\sqrt{3}$

C 18

D $18\sqrt{3}$

E 36

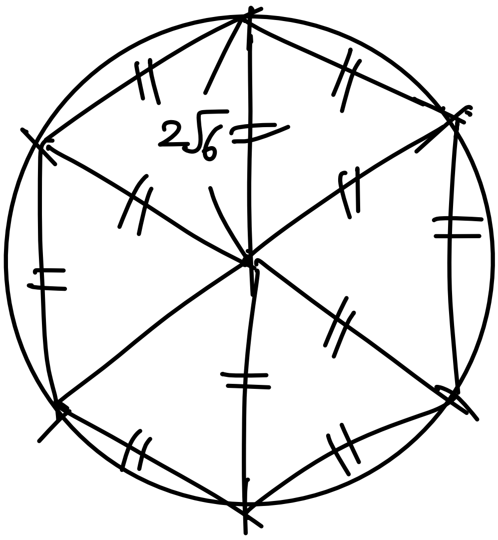
F $36\sqrt{3}$

G 48

H $48\sqrt{3}$

$$(x-9)^2+(y-11)^2=-178+9^2+11^2$$

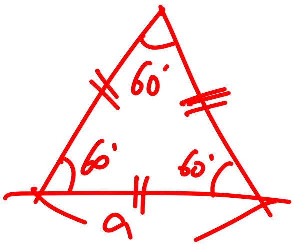
$$(x-9)^2+(y-11)^2=24=(2\sqrt{6})^2$$



$$\frac{\sqrt{3}}{4}\times(2\sqrt{6})^2\times 6$$

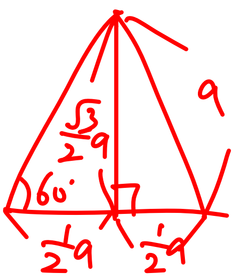
$$=\frac{\sqrt{3}}{4}\times 4\times 6\times 6$$

$$=36\sqrt{3}.$$



\# $S_{\text{equilateral triangle}} = \frac{\sqrt{3}}{4}a^2$

Proof)



$a\times\frac{\sqrt{3}}{2}a\times\frac{1}{2}=\frac{\sqrt{3}}{4}a^2$ //

$\star$ if l_1 & l_2 are perpendicular, the product of their gradients are -1.

e.g.) $y = x$ & $y = -x$.

tangent

$f(x)$: Gradient (tangent) $\times$ Gradient (normal) $= -1$.

P

10 A curve C has equation $y = f(x)$ where

normal

$$f(x) = p^3 - 6p^2x + 3px^2 - x^3$$

and p is real. $f'(x) = -6p^2 + 6px - 3x^2$.

The gradient of the normal to the curve C at the point where $x = -1$ is M.

What is the greatest possible value of M as p varies?

A $-\frac{3}{2}$

B $-\frac{2}{3}$

C $-\frac{1}{2}$

D $\frac{1}{4}$

E $\frac{2}{3}$

F $\frac{3}{2}$

sol) : $\underbrace{\dfrac{-1}{f'(-1)}}(x-(-1)) + f(-1)$

$\rightarrow \mathrm{Max}(M) = \mathrm{Max}\left(\dfrac{-1}{f'(-1)}\right)$

$f'(-1) = -6p^2 - 6p - 3$.

$\mathrm{Max}\left(\dfrac{-1}{-6p^2-6p-3}\right) = \mathrm{Max}\left(\dfrac{1}{6p^2+6p+3}\right)$

$= \mathrm{Min}(\underbrace{6p^2+6p+3})$.

$\Rightarrow 6(p^2+p+\frac{1}{4}) + \frac{3}{2}$

$= 6(p+\frac{1}{2})^2 + \frac{3}{2}$ //

$\star$ Gradient to tangent at $y = f(x)$ at P is $f'(p)$

normal at at $P(p, f(p))$ is $\dfrac{-1}{f'(p)}$. $y = f(x)$.

☆ General rule to sequences. Jot 'em.
Laying out terms, patterns USUALLY show up.

11 The sequence x_n is defined by the rules

$$x_1 = 7$$

$$x_{n+1} = \frac{23x_n - 53}{5x_n + 1}$$

The first three terms in the sequence are 7, 3, 1

What is the value of x_{100}?

A -5

B 0

C 1

D 3

E 7

$x_1 = 7$

$x_2 = 3$

$x_3 = 1.$

$n=3: \; x_4 = \frac{23x_3 - 53}{5x_3 + 1} = \frac{23-53}{5+1} = -5$

$n=4: \; x_5 = \frac{23x_4 - 53}{5x_4 + 1} = \frac{23(-5) - 53}{5(-5)+1}$

$= \frac{-168}{-24} = 7$ ✓

then at

$n=5: \; x_6 = \frac{23x_5 - 53}{5x_5 + 1}$ would be 3.

Cyclic sequence.

when cycle is k, and finding a_n, find remainder when dividing n by k.

$(7 \; 3 \; 1 \; -5)(7 \; 3 \; 1 \; -5)$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8$

$\# \int_{a+k}^{b+k} f(x)\,dx = \int_a^b f(x+k)\,dx$, $\int_a^b f(x)\,dx = \int_{a-k}^{b-k} f(x+k)\,dx$

12 The polynomial function $f(x)$ is such that $f(x) > 0$ for all values of x.

Given $\int_2^4 f(x)\,dx = A$, which one of the following statements **must** be correct?

A $\int_0^2 [f(x+2)+1]\,dx = A+1$

B $\int_0^2 [f(x+2)+1]\,dx = A+2$

C $\int_2^4 [f(x+2)+1]\,dx = A+1$

D $\int_2^4 [f(x+2)+1]\,dx = A+2$

E $\int_4^6 [f(x+2)+1]\,dx = A+1$

F $\int_4^6 [f(x+2)+1]\,dx = A+2$

sol) $\int_2^4 f(x)\,dx$

$= \int_{2-2}^{4-2} f(x+2)\,dx$

$= \int_0^2 f(x+2)\,dx$

$\int_0^2 [f(x+2)+1]\,dx$

$= \int_0^2 f(x+2)\,dx + \int_0^2 1\,dx$

$= A + [x]_0^2 = A+2-0 = A+2$

Proof) $\int_a^b f(x+k)\,dx$. let $x+k=t$

$dx = dt$

$= \int_{a+k}^{b+k} f(t)\,dt$

Make a habit of translating question straight into one simple formula.

13 In the expansion of $(a + bx)^5$ the coefficient of x^4 is 8 times the coefficient of x^2.

Given that a and b are non-zero **positive** integers, what is the smallest possible value of $a + b$?

A 3

B 4

C 5

D 9

E 13

F 17

$$\frac{\binom{5}{4}\cancel{b^4}\,\cancel{a}\,b^2}{\binom{5}{2}\cancel{b^2}\,\cancel{a^3}\,a^2} = 8$$

$$\frac{\cancel{5}b^2}{2\cancel{10}a^2} = 8$$

$$b^2 = 16a^2.$$

$b = 4a$ (because $b>0$. $a>0$).

$a = 1. 2. 3$ ...

$\min(a+b) = \min(5a) = 5.$

find correlations between given equations
Don't see them discretely. Harmonise them.

14 The solution of the simultaneous equations

$$2^x + 3 \times 2^y = 3 \quad \cdots \text{①}$$

$$2^{2x} - 9 \times 2^{2y} = 6 \quad \cdots \text{②}.$$

is $x = p,\ y = q$.

Find the value of $p - q$

A $\frac{5}{12}$

B $\frac{7}{3}$

C $\log_2 \frac{5}{12}$

D $\log_2 \frac{7}{3}$

E $\log_2 9$

F $\log_2 15$

$$2^x + 3\times 2^y = 3$$

$$\times\ (2^x - 3\times 2^y) = 2.$$

$$2^{2x} - 9\times 2^{2y} = 6.$$

$$\frac{②}{①} : 2^x - 3\times 2^y = 2 \quad \cdots ③$$

$$① + ③ : 2\times 2^x = 5$$

$$2^x = \frac{5}{2}, \quad x = \log_2\left(\frac{5}{2}\right)$$

$$① - ③ : 6\times 2^y = 1$$

$$2^y = \frac{1}{6}, \quad y = \log_2\left(\frac{1}{6}\right)$$

$$p - q = x - y = \log_2\left(\frac{5}{2}\right) - \log_2\left(\frac{1}{6}\right)$$

$$= \log_2 15 \ //$$

None of checked numbers matter, period. Don't waste time trying to find exact position on coordinates.

15 It is given that $f(x) = -2x^2 + 10$

for trapezium Rule, only check concavity.

Consider the following three curves:

(**1**) $y = f(x)$

(**2**) $y = f(x+1)$

(**3**) the curve $y = f(x+1)$ reflected in the line $y = 6$

reflected by a horizontal line

The ~~trapezium rule is~~ used to estimate the area under each of these three curves between $x = 0$ and $x = 1$.

State whether the trapezium rule gives an overestimate or underestimate for each of these areas.

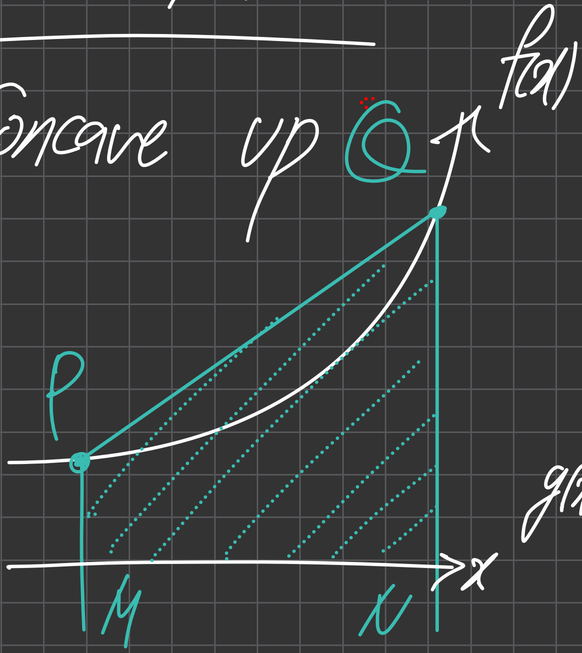
☆☆ Concave Up (∪): Overestimate, Concave Down (∩): Underestimate

	(**1**)	(**2**)	(**3**)
A	underestimate	underestimate	underestimate
B	underestimate	underestimate	overestimate
C	underestimate	overestimate	underestimate
D	underestimate	overestimate	overestimate
E	overestimate	underestimate	underestimate
F	overestimate	underestimate	overestimate
G	overestimate	overestimate	underestimate
H	overestimate	overestimate	overestimate

Don't differentiate it twice & compare signs... I mean it's not wrong but concavity is a graph property and if you can draw f(x), in this case you can, do so.

Trapezium Rule.

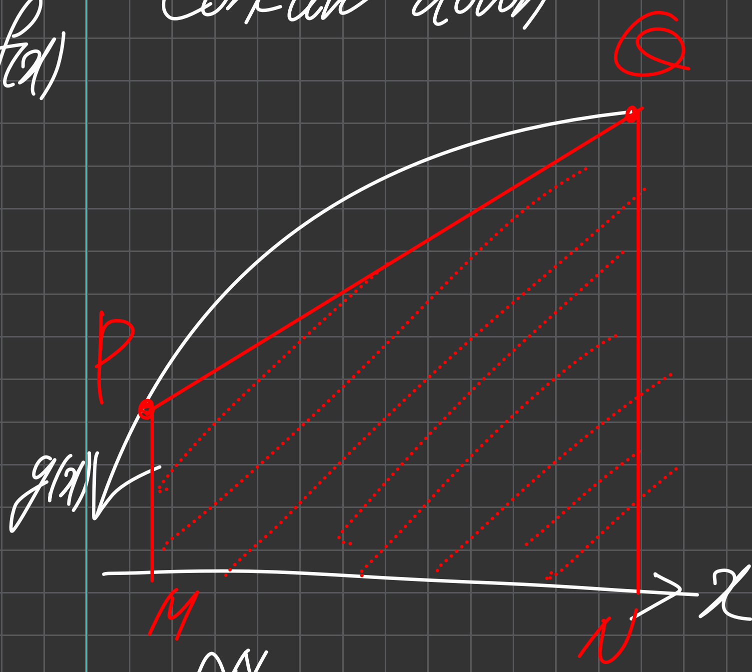
Concave up



$\int_{\square MNQP} > \int_M^N f(x)\,dx$

Over estimate

Concave down



$\int_M^N g(x)\,dx > \int_{\square MNQP}$

under estimate

Only Concave/ Convex matters. No need to exactly plot Graph, this is pure waste of time.

Just like Q5.
Think in intersections & unions (thus sets) for inequalities

16 The functions f and g are given by $f(x) = 3x^2 + 12x + 4$ and $g(x) = x^3 + 6x^2 + 9x - 8$.

What is the complete set of values of x for which one of the functions is increasing and the other decreasing?

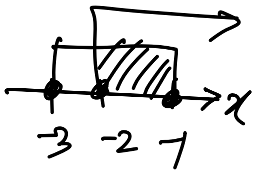
$f' \geq 0 \cap g' \leq 0.$

i) f increases $\cap$ g decreases

$f'(x) = 6x+12 = 6(x+2) : x \geq -2.$

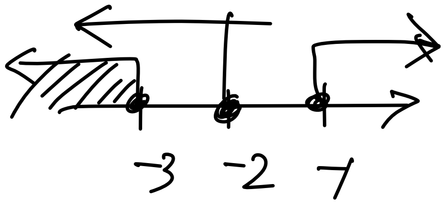
$g'(x) = 3x^2+12x+9$

$= 3(x+1)(x+3) : -3 \leq x \leq -1.$



ii) f decreases $\cap$ g increases

$\{x \leq -2\} \cap \{x \leq -3. \text{ and } x \geq -1\}.$



A $x \geq -1$

B $x \leq -1$

C $-3 \leq x \leq -2, \; x \geq -1$

D $x \leq -2, \; x \geq -1$

E $x \leq -3, \; -2 \leq x \leq -1$

F $x \leq -3, \; x \geq -2$

G $-2 \leq x \leq -1$

Know what each variable is.

$n \in \mathbb{Z}$

$n > 0$

17 The two functions $F(n)$ and $G(n)$ are defined as follows for positive integers n:

D

$$F(n) = \frac{1}{n}\int_0^n (n-x)\,dx$$

$$G(n) = \sum_{r=1}^{n} F(r)$$

What is the smallest positive integer n such that $G(n) > 150$?

A 22

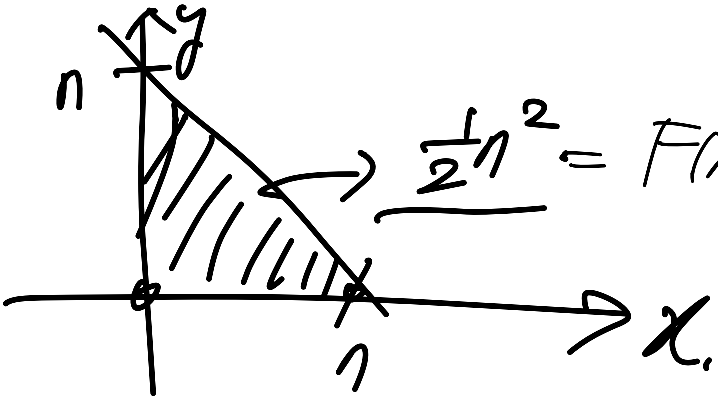
B 23

C 24

D 25

E 26

sol)



$\frac{1}{2}n^2 = F(n)$

$F(n) = \frac{1}{2}n^2 \times \frac{1}{n} = \frac{1}{2}n$

$G(n) = \frac{1}{2}\sum_{r=1}^{n} r$

$= \frac{1}{2} \times \frac{n(n+1)}{2}$

$= \frac{1}{4}n(n+1)$

$\frac{1}{4}n(n+1) > 150$

try values.

$n = 24+1$ $n(n+1) > 600.$

$25 \times 24 = 600$

$(25 \times 4) \times 6$

Beware of conflating kx & $\frac{x}{k}$.

18 The graph of $y = \log_{10}x$ is translated in the positive y-direction by 2 units.

This translation is equivalent to a stretch of factor k parallel to the x-axis.

What is the value of k?

A 0.01

B $\log_{10} 2$

C 0.5

D 2

E $\log_2 10$

F 100

$y - 2 = \log_{10} x.$

$y = 2 + \log_{10}(x)$

$= \log_{10}(100x)$

$y = \log_{10}\left(\frac{1}{k}x\right).$

$\Leftrightarrow$

$k = \frac{1}{100}$

$\boxed{\text{And}} = \boxed{\cap}$, $\boxed{\text{Or}} = \boxed{\cup}$ (Union)

intersection

$p < q \cdots ①$

19 The set of solutions to the inequality $x^2 + bx + c < 0$ is the interval $p < x < q$ where b, c, p and q are real constants with $c < 0$.

Best way to think of intersections are in sets.

In terms of p, q and c, what is the set of solutions to the inequality $x^2 + bcx + c^3 < 0$?

A $\frac{p}{c} < x < \frac{q}{c}$

B $\frac{q}{c} < x < \frac{p}{c}$

C $pc < x < qc$

trick.

D $qc < x < pc$

E $pc^2 < x < qc^2$

F $qc^2 < x < pc^2$

$x^2 + bx + c = (x-p)(x-q)$

$= x^2 - (p+q)x + pq$

$b = -(p+q)$

$c = pq < 0 \quad \cdots ②$

$p < q$ and $pq < 0$

so $p < 0 < q$.

$x^2 + bcx + c^3 < 0$

$= x^2 - pq(p+q)x + p^3q^3$

$= (x - pq^2)(x - p^2q)$

$pq = c$ substitute

$(x - qc)(x - pc) < 0$

As $p < 0 < q$ and $c < 0$, ($p = \ominus$, $q = \oplus$, $c = \ominus$, so $pc = \oplus$, $qc = \ominus$)

$pc > qc$.

$\therefore \; qc < x < pc$

$(x-x_1)(x-x_2)(x-x_3)$ $= a_1x^3 - a_1(a_1+a_2+a_3)x^2 +$

$a_1(a_1a_2+a_2a_3+a_3a_1)x + a_1a_2a_3$ a_1

a_1

$= a_1x^3 + a_2x^2 + a_3x^1 + a_4 = 0.$

$-\frac{a_2}{a_1} = x_1+x_2+x_3.$

$+\frac{a_3}{a_1} = x_1x_2+x_2x_3+x_3x_1.$

$-\frac{a_4}{a_1} = x_1x_2x_3.$

$1+2+3=6.$

$f(x) = (x-1)(x-2)(x-3).$

$= x^3-6x^2+11x-6.$

$\left(2-\frac{1}{\sqrt3}\right) + \left(2-\frac{1}{\sqrt3}\right) + ☆ = 6.$

$l = mx+n.$

$f(x) = l$

$g(x) = f(x) - l = 0.$

$\left(2+\frac{1}{\sqrt3}\right)\times 2 + △ = 6.$

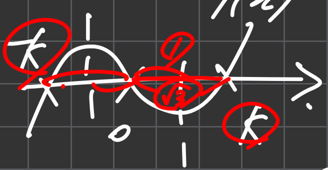
$x^3-6x^2+11x-6-mx-n.$

$g(x) = x^3-6x^2+(11-m)x-n-6=0.$

$g(x)=0$ $\alpha\ \beta\ \gamma.$

$\alpha+\beta+\gamma = -\left(\frac{-6}{1}\right) = 6$

$-\frac{k}{\sqrt3}$



$\frac{k}{\sqrt3}$

$f(x) = x(x-k)(x+k)$

$= x^3-k^2x.$

$f'(x) = 3x^2-k^2$

$= (\sqrt3x-k)(\sqrt3x+k).$

$\frac{k}{\sqrt3} : k$

$\frac{1}{\sqrt3} : 1$

$1:\sqrt3.$

<Vieta's formula>

know at least up to quartics (4th degree at max).

Any degree of polynomial of degree n, let

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{\star} x^{n-\star} + \cdots + a_{n-1} x^1 + a_n x^0$$

let roots of $P(x) = 0$ be

$x = r_1, r_2, \cdots r_n$

Vieta's formulae are,

$$\sum_{k=1}^{n} r_k = \ominus \frac{a_1}{a_0}$$

$$(r_1 r_2 + r_1 r_3 + \cdots r_1 r_{n-1} + r_1 r_n) + (r_2 r_3 + r_2 r_4 + \cdots + r_2 r_n) \cdots + (r_{n-1} r_n) = \oplus \frac{a_2}{a_0}$$

$$\vdots$$

$$r_1 r_2 \cdots r_n = (-1)^n \frac{a_n}{a_0}.$$

(Proof) $P(x) = ax^n + \cdots$

(i) $n=2$

$P(x) = ax^2 + bx + C$. formulae: $r_1 + r_2 = -\frac{b}{a}$, $r_1 \times r_2 = \frac{c}{a}$

let $P(x)=0$ have roots of $x = r_1, r_2$

$P(x) = a(x-r_1)(x-r_2)$ ↓ Binomial Expansion

$= a(x^2 - (r_1+r_2)x + r_1r_2)$

$= ax^2 \underbrace{- a(r_1+r_2)}_{=b} x + \underbrace{ar_1r_2}_{=C}$

$-\frac{b}{a} = -\frac{-a(r_1+r_2)}{a} = r_1 + r_2 = \sum_{k=1}^{2} r_k$

$+\frac{c}{a} = \frac{a(r_1r_2)}{a} = r_1r_2 = \prod_{k=1}^{2} r_k$ ($\prod_{k=1}^{n} a_k$ just means $a_1 \times a_2 \times \cdots \times a_n$)

Hence Vieta's formulae is true for $n=2$.

mnemonics: start with $\ominus$, then alter signs always.

(ii) $n=3$

$P(x) = ax^3 + bx^2 + Cx + d$. formulae:

$-\frac{b}{a} = r_1 + r_2 + r_3$

$+\frac{c}{a} = r_1r_2 + r_2r_3 + r_3r_1$

$-\frac{d}{a} = r_1r_2r_3$

proof)

let $P(x)=0$ have $x = r_1, r_2, r_3$

$P(x) = a(x-r_1)(x-r_2)(x-r_3)$

$= ax^3 + bx^2 + Cx + d$.

Expand & compare coefficient.

TL;DR,

Try it yourself.

See it in play.

iii) Quartic: $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

let $P(x) = 0$ have roots of

$x = r_1, r_2, r_3, r_4$ (Do not even need to be real roots).

$$P(x) = a(x-r_1)(x-r_2)(x-r_3)(x-r_4)$$

6 elements. Why? $4C_2 = \frac{4!}{2!2!} = 6$.

Expanding will give...

$4C_1 = 4$ elements.

sum of all roots

sum of the product of two roots

$$P(x) = ax^4 - a(r_1 + r_2 + r_3 + r_4)x^3 + a(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2$$
$$- (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)x + r_1r_2r_3r_4$$

sum of the products of 3 roots.

4 elements here. Why? $4C_3 = \frac{4!}{1!3!} = 4$.

sum of the product of all 4 roots (so only one)

only one element. $4C_4 = 1$.

notice how sign alters.

Hence, $r_1 + \cdots + r_4 = -\frac{b}{a}$ *

$r_1r_2 + r_1r_3 + \cdots + r_3r_4 = +\frac{c}{a}$

$r_1r_2r_3 + \cdots + r_2r_3r_4 = -\frac{d}{a}$

$r_1r_2r_3r_4 = +\frac{e}{a}$

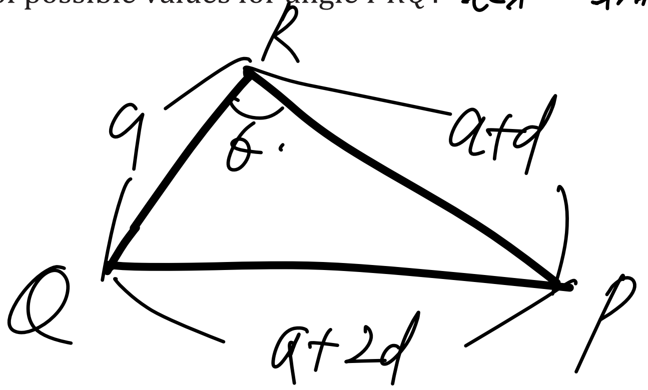
signs alter, starting at $-$.

Reaction Circuit (Asked by Muskaan) When 3 sides of all triangles are given, the angle of interest is using cosine rule.

Killer. Getting this right is your edge.

\# The lengths of the sides QR, RP and PQ in triangle PQR are a, $a+d$ and $a+2d$ respectively, where a and d are positive and such that $3d > 2a$. $a>0$ so $\frac{3d}{2a}>1$. let this be θ.

What is the full range, in degrees, of possible values for angle PRQ?



A $0 <$ angle $PRQ < 60$

B $0 <$ angle $PRQ < 120$

C $60 <$ angle $PRQ < 120$

D $60 <$ angle $PRQ < 180$

E $120 <$ angle $PRQ < 180$

Basic property of triangle:

$(a)+(a+d)>(a+2d)$

$a>d$

$\frac{d}{a}>1$

\# Don't skip this, you got lucky because in this specific question it's irrelevant.

$$\cos\theta = \frac{(a)^2+(a+d)^2-(a+2d)^2}{2a(a+d)}$$

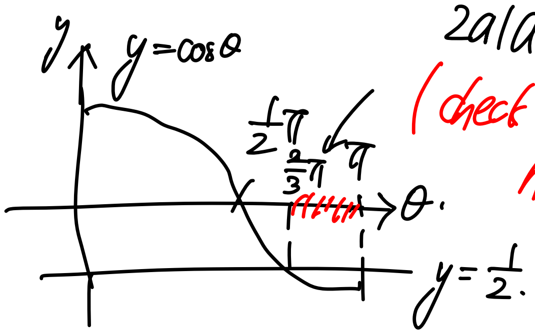
$$= \frac{a^2+\cancel{a^2}+2ad+d^2-\cancel{a^2}-4ad-4d^2}{2a(a+d)}$$

$$= \frac{a^2-2ad-3d^2}{2a(a+d)} = \frac{\cancel{(a+d)}(a-3d)}{2a\cancel{(a+d)}} = \frac{1}{2}-\frac{3d}{2a}$$

$$-\frac{3d}{2a}<-1$$

$$\cos\theta = \frac{1}{2}-\frac{3d}{2a}<-\frac{1}{2}$$

simply derived

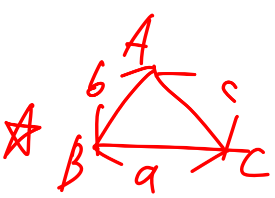


(check Question #8 notes)

$$\cos(\angle A) = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos(\angle B) = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos(\angle C) = \frac{a^2+b^2-c^2}{2ab}$$



from ($a^2=b^2+c^2-2bc\cos\angle A$ & leaving $\cos\angle A$ on LHS)

END OF TEST

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